1.6 Relations and Functions

STEM professionals working in industry, government or academia often try to identify or create relationships between two different types of quantitative data. For example, digital cameras relate the intensity of light to voltages in an electric circuit. Linear Algebra provides elegant theoretic results about linear functions between two vector spaces. Because many relationships that exist in the real-world can be simplified into linear functions, we can apply the tools of linear algebra to solve real-world problems

When studying real-world problem, we often collect quantitative data. Suppose we measure quantity a from an experiment related to one aspect of our problem. At same time, we measure a different quantity b. We use an ordered pair (a, b) to indicate that a is related to b. This framework sets the foundation for the creation of very general relations between any objects. We begin our study of relations by defining the set of all possible combinations of data taken from two different sets.

Definition 1.4: Cross Product of Sets

Let A and B be sets. The **cross product** of A and B is the set

$$A \times B = \{(a, b) \,:\, a \in A \text{ and } b \in B\}.$$

The notation $A \times B$ is read "A cross B."

Elements of the cross product of sets take the form (a, b), known as an **ordered-pair** formed from elements $a \in A$ and $b \in B$. The ordered pair (a, b) has two coordinates written inside the parenthesis: the **first coordinate** a written on the left of the comma and the **second coordinate** b written on the right of the comma. We say two ordered pairs (a, b) and (x, y) are **equal** iff $a = x$ and $b = y$. Changing either coordinate of a given ordered pair (a, b) yields a different ordered pair.

The cross product $A \times B$ is the set of all possible ordered pairs that can be made by choosing first coordinates from the set A and second coordinates from the set B. We can also refer to the cross product as the Cartesian Product of two sets. Be very careful: the cross product between two sets is different from the cross product between vectors that you studied in calculus.

EXAMPLE 1.6.1

Suppose we are studying the following real-world problem: How do campaign contributions from corporate donors affect the voting record of US Senators on corporate sponsored bill? In order to study this problem, we will map voting records into the space of numbers. To consider all the possible ways to do this, we look at the cross product between sets $A = \{-1, 0, 1\}$ and $B = \{\text{Nay}, \text{Abstain}, \text{Yay}\}$. Then

$$\begin{aligned} A \times B = \{&(-1, \text{Nay}), (-1, \text{Abstain}), (-1, \text{Yay}),\\ &(0, \text{Nay}), (0, \text{Abstain}), (0, \text{Yay}),\\ &(1, \text{Nay}), (1, \text{Abstain}), (1, \text{Yay}), \}. \end{aligned}$$

This cross product gives an exhaustive list of all possible ways to combine the numbers $-1, 0, 1$, with the voting record of any senator on one bill.

EXAMPLE 1.6.2

Imagine we were some of the first humans to come up with a system to organize the alphabet. In particular, imagine we are creating the first dictionary ever made. Recall the set U of upper case english letter from Example 1.5.1. We will create our ordering of elements in U by relating the upper case letters to numbers in the set

$$[26] = \{x \in \mathbb{N} : 1 \leq x \leq 26\}$$

We begin our study with a brute force (and naive) approach of listing all possible ordered pairs in the set $U \times [26]$. This set has 26^2 different ordered pairs in the form (Ω, n), where $\Omega \in U$ and $n \in [26]$. This cross product is the set of all possible ways to combine capital letters with the natural numbers from 1 to 26.

We can generalize the definition of the cross product of two sets. In particular, the **cross product of three sets** A, B, C is defined as the set of all order triplets:

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}.$$

For $n \in N$, we define the **cross product of** n **sets**, $A_1, A_2, ..., A_n$, as

$$A_1 \times A_2 \times \cdots A_n = \{(a_1, a_2, ..., a_n) : a_i \in A_i \text{ for all } i \in \{1, 2, ..., n\}\}.$$

Each element of this cross product is known as an **ordered n-tuple**. Two ordered n-tuples $(a_1, a_2, ..., a_n)$ and $(x_1, x_2, ..., x_n)$ are equal if and only if $a_i = x_i$ for every $i = 1, 2, ..., n$.

EXAMPLE 1.6.3

In Vector Calculus, we study functions whose domain is a subset of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and whose range is contained in $\mathbb{R}$. The set

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) : x, y, z \in \mathbb{R}\}.$$

This is the set of all possible ways to form a $3-$tuple, where each coordinate is some real number.

The cross product is a formal mathematical tool used to discuss the set of all possible combinations that can be made when creating order pairs. Most of the time, we don't want to consider the entire set of combinations, since this often is not useful in the modeling process. Instead, when creating useful mathematical models, we focus on a select few ordered pairs that encode relationships between data that we've collected on our problem. From this perspective, the purpose of defining the cross product of sets is to establish the set of all possible choices from which more specific combinations will be chosen.

Definition 1.5: Relation

Let A and B be sets. The set R is a **relation from** A **to** B iff

$$R \subseteq A \times B.$$

A relation from A to itself is called **a relation on A**. If $(a, b) \in R$, we write $\mathbf{a\,R\,b}$ and say that a is **related** to b. If $(a, b) \notin R$, then $\mathbf{a\,\not{R}\,b}$.

EXAMPLE 1.6.4

Suppose we are designing a traffic control systems. We want to know: How can we use symbols to direct traffic? To address this problem, we introduce a relevant mathematical relation. Let $A = \{\textcolor{red}{\bullet}, \textcolor{green}{\bullet}\}$ be the set of red and green colored circles and $B = \{\text{STOP}, \text{GO}\}$ be the set of actions desired for drivers. Then

$$A \times B = \{(\textcolor{red}{\bullet}, \text{STOP}), (\textcolor{red}{\bullet}, \text{GO}), (\textcolor{green}{\bullet}, \text{STOP}), (\textcolor{green}{\bullet}, \text{GO})\}$$

We decide on the relation

$$R = \{(\textcolor{red}{\bullet}, \text{STOP}), (\textcolor{green}{\bullet}, \text{GO})\}$$

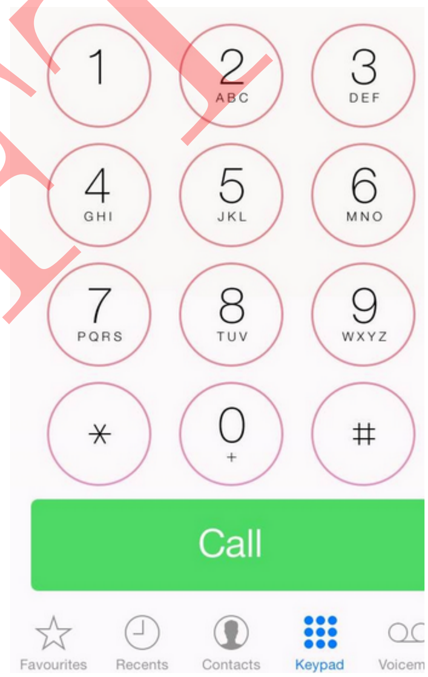
Notice that $R \subset A \times B$. In this case, we've used the the set theoretic version for relations to establish a solution to our traffic control problem.

EXAMPLE 1.6.5

The touch-tone telephone is well known in the United States. Prior to the existence of Smart Phones, a physical telephone that contain this technology was a ubiquitous part of the US telecommunications infrastructure. The dial pad on many of today's smart phones contains a digital image of the same interface. Regardless of whether the faceplate is a physical or digital object, each button indicates special properties. The relation implicit in this technology is as follows:

$$D \subseteq N \times U,$$
$$D = \{(2, \text{A}), (2, \text{B}), (2, \text{C}), (3, \text{D}), (3, \text{E}), (3, \text{F}), ..., (9, \text{Y}), (9, \text{Z})\}.$$

where U is the set of upper case english letters discussed in Example 1.5.1 and $N = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Notice, there are 26 different elements in R. Also notice that $(1, \text{G}) \notin D$ and thus D is a proper subset of $N \times U$.



This image is taken from an iOS d pad. Notice the relation implicit this technology.

EXAMPLE 1.6.6

Consider the real-world problem: How can we design a machine to perform calculations between numbers? The first generation of scientists to focus on this problem realized that all data could be encoded using a two letter alphabet,$\{0, 1\}$. The symbol 0 is called logic 0 and the symbol 1 is known as logic 1 (for more about this, Google search"Boolean Algebra"). Claude Shannon, the father of Information Theory and the first person to draw connections between formal mathematical logic and electric circuits, realized that electricity could be used encode the value of a two letter alphabet into a machine. In particular, given a high voltage H, and a Low voltage L, an explicit relation could be established between the fundamental unit of encoding and physical measurements of electric current. For example, we could define a relation, known as the active high relation where

$$A_H = \{(0, L), (1, H)\}.$$

This corresponds with a digital computer in which high voltages stored in the computer's hardware are interpreted as logic 1 while low voltages are interpreted as logic 0. On the other hand, the relation known as active low has the opposite orientation:

$$A_L = \{(0, H), (1, L)\}.$$

In this case, high voltages correspond to logic 0 while low voltages to logic 1.

It is worth noting that the actual physical values of H and L vary greatly depending on the fabrication process of the digital hardware. For example, the Texas Instrument TTL (Transistor-Transistor Logic) family has a H value of 3.3 Volts and a L value of 0.5 Volts.

This general set theoretic definition of a relation is powerful because we can state any type of correspondence in terms of a set of ordered pairs. By requiring that a relations is any subset of a cross product, we guarantee that any set of ordered pairs is a relation. When discussing relations, it is very helpful to have names for the sets that comprise the first- and second-coordinates of the ordered pairs in that relation.

Definition 1.6: Domain, Range and More

Let A and B be sets and let R be a relation from A to B. The **domain space** of relation R is the set A. The **domain** of the relation R is the set

$$\text{Dom }(R) = \{x \in A : \text{ there is a } y \in B \text{ such that } (x, y) \in R\}.$$

The **codomain** of the relation R is the set

$$\text{Codom }(R) = B.$$

The **range** of the relation R is the set

$$\text{Rng }(R) = \{y \in B : \text{ there is a } x \in A \text{ such that } (x, y) \in R\}.$$

The domain space of R is the set from which first coordinates of any ordered pair in R are chosen. The domain of relation R is the set of all first coordinates of the ordered pairs in R. The codomain of R is the set from which the second coordinates of any ordered pair $(a, b) \in R$ are chosen. The ranges of R is the set of all second coordinates of the ordered pairs in R.

EXAMPLE 1.6.7

Looking back at the "Dial Pad" relation from Example 1.6.5 we see

$$\begin{aligned} \text{Domain Space}(D) &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, \\ \text{Dom}(D) &= \{2, 3, 4, 5, 6, 7, 8, 9\}, \\ \text{Codom}(D) &= U, \\ \text{Rng}(D) &= U. \end{aligned}$$

In this case, U represents the set of upper case letters in the English alphabet, as defined in Example 1.5.1.

EXAMPLE 1.6.8

Consider the following relation:

$$E = \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} : \frac{x^2}{256} + \frac{y^2}{25} \leq 1 \right\}.$$

If we graph this relation using the Cartesian plane, we see an ellipse. Moreover, we can find

$$\begin{aligned} \text{Domain Space}(E) &= \mathbb{R} \\ \text{Dom}(D) &= [-16, 16] \subset \mathbb{R} \\ \text{Codom}(D) &= \mathbb{R}, \\ \text{Rng}(D) &= [-5, 5] \subset \mathbb{R}. \end{aligned}$$

EXAMPLE 1.6.9

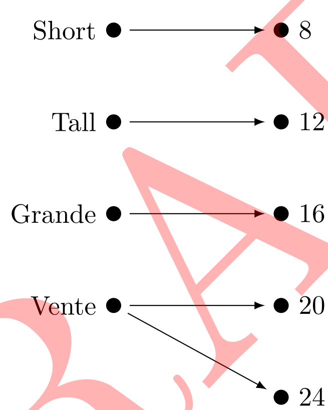
Let's consider the relation that Starbucks Coffee defines between the names for the sizes of the drinks and the the volume of a Starbucks drink. Below we list this relation completely:

$$\{(\text{Short}, 8), (\text{Tall}, 12), (\text{Grande}, 16), (\text{Vente}, 20), (\text{Vente}, 24)\}$$

There are a number of ways to visualize a relation. For example, with the element enumeration method, we list all the elements of the ordered pair as seen above. We could also list the elements of the relation in a two-column table.

Size Name	Fluid Ounces
Short	8
Tall	12
Grande	16
Vente	20
Vente	24

Yet a third method would be to display the relation using an arrow diagram.



In an arrow diagram, each unique element in the domain is represented by a single, labeled dot. Similarly each unique element in the range is represented by a single, labeled dot. Arrows are drawn from domain elements to range elements if and only if the corresponding ordered pair is found in the relation.

We now construct a formal definition of a function by refining the concept of a relation. Recall that a function is a special type of correspondence where by each input value (each element in the domain) corresponds to a unique output value (one and only one element in the Codomain). The following definition provides a set theoretic definition of a function.

Definition 1.7: Function

A **function from A to B** is a relation f from A to B such that both of the following hold

i. Dom $(f) = A$

ii. if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

We denote the phrase "f is a function from A to B" with the notation $f : A \to B$. If $B = A$, we say that f is a **function on** A.

From our definition above, we see that if $f : A \to B$, then $f \subseteq A \times B$. In other words, all functions are relations. However, not all relations are functions.

EXAMPLE 1.6.10

One of the most famous functions used in computer science is given by a mapping between the set of nonnegative integers between 0 and $2^k - 1$ for some $k \in \mathbb{N}$ and the set of k digit binary integers. For example, we see can write the decimal number 6 as a 3−bit binary integer using the following realization:

$$6 = 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 110$$

We can also convert the decimal integer 13 into a 4−bit binary integer using the map suggested above:

$$13 = 8 + 4 + 1 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 1101.$$

In other words, if

$$\begin{aligned} N_k &= \{n \in \mathbb{Z} \,:\, 0 \leq n \leq 2^k - 1\}, \\ B_k &= \{b_k b_{k-1} \cdots b_2 b_1 \,:\, b_i = 0 \text{ or } b_i = 1 \text{ for each } i = 1, 2, ..., k\}. \end{aligned}$$

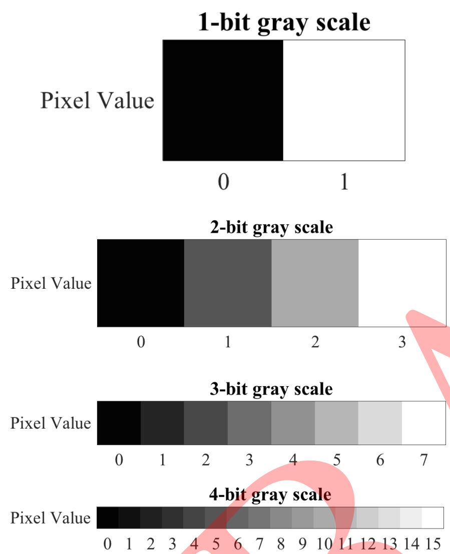
computer scientist establish the binary representation of a number as a subset of $N_k \times B_k$ where the map is defined as

$$n = b_k \cdot 2^{k-1} + b_{k-1} \cdot 2^{k-2} + \cdots + b_2 \cdot 2^1 + b_1 2^0.$$

EXAMPLE 1.6.11

When you take a black and white picture with a digital camera, you capture analog information (a collection of fluctuating light waves in the frame of your picture) and transform these waves into digital information (a collection of binary digits inside a computer). The technology inside the camera focuses light from your camera lens onto a digital image sensor. These sensors provide a grid of tiny *photosites*, each one called a *pixel*. Each pixel is a light-sensitive electronic device that converts photons from incoming light into an analog voltage level that can then be digitized. Once this digital information is stored, your camera must have a way to convert the digital information of a picture into an image that you can recognize.

In this example, we study the *gray scale*. The gray scale is a standard color model that indicates exactly how binary numbers are translated into shades of gray corresponding to luminous intensities captured at each pixel. To begin our study, we note that for each pixel, our camera has set a color depth (also known as bit depth). This is the number of bits dedicated to each pixel. Below, we show gray scales corresponding to $1-$, $2-$, $3-$ and $4-$ bit depth models.



We can list all of the possible representations of light using our knowledge of binary representations. Once we've done so, we should determine the mapping between each binary number and the corresponding intensity of light. Below is a fictitious example of one way to do this for a black and white image:

In the above diagrams, we've presented the stored binary number in decimal form. However, when these values are stored in a digital camera's memory, these exist as voltage values representing the corresponding binary numbers. A $4-$bit gray scale indicates that each pixel has 4 different voltage values used to store the intensity of light sensed at that pixel. For more about this, see the wikipedia articles on "digital camera," "color model," "color depth" and "luminous intensity."

There are a number of very popular sets of functions that show up in numerous fields in mathematics. These sets of functions are used in many of the applications we present in this text. Below we define each of these for future reference:

Definition 1.8: Important Sets of Functions

Let $I \subseteq \mathbb{R}$. Then

$$\begin{aligned}
P_n(I) &= \{f \,:\, f : I \to \mathbb{R} \text{ and } f \text{ is a polynomial with } \deg(f) \leq n \text{ for } n \in \mathbb{N}\}. \\
C^{(\infty)}(I) &= \{f \,:\, f : I \to \mathbb{R} \text{ and } f \text{ has continuous derivatives of all orders on } I\}. \\
C^{(p)}(I) &= \{f \,:\, f : I \to \mathbb{R} \text{ and } f \text{ has continuous } p\text{th derivative for } p \in \mathbb{N} \text{ on } I\}. \\
C^{(1)}(I) &= \{f \,:\, f : I \to \mathbb{R} \text{ and } f \text{ has continuous first derivative on } I\}. \\
C(I) &= \{f \,:\, f : I \to \mathbb{R} \text{ and } f \text{ is a continuous on } I\} \\
F(I) &= \{f \,:\, f : I \to \mathbb{R}\}.
\end{aligned}$$

EXAMPLE 1.6.12

On the other can, if we want to define the set of continuous functions on the closed interval $[-1, 1]$, we might declare

$$C([-1,1]) = \{f \,:\, f \text{ is a continuous function with domain } [-1,1]\}.$$

Again, we choose our set name to be suggestive its significance and we use the variable name f since many readers will probably associate functions with this variable.

EXAMPLE 1.6.13

Let $C^{(1)}([-1,1])$ be the set of continuous, differentiable, real-valued functions defined on the interval $[-1,1]$ whose derivatives are also continuous on that interval. Let $C([-1,1])$ be the set of continuous functions. We want to show that $C^{(1)}([-1,1]) \subseteq C([-1,1])$.

Suppose that $f \in C^{(1)}([-1,1])$.
To prove that f is continuous, we need to show that $\lim_{x\to a} f(x) = f(a)$ for all $a \in [-1,1]$. We will establish this fact by showing that the difference between $f(x)$ and $f(a)$ goes to zero as x approaches a.

Suppose that $a \in [-1,1]$. Since $f(x)$ is differentiable at a, we know that

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists. Moreover, assuming that $x \neq a$, we know that

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a).$$

Because the limit of a product is equal to the product of the limits (see Appendix B for more), we know

$$\begin{aligned}
\lim_{x\to a} [f(x) - f(a)] &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a), \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a}(x - a), \\
&= f'(a) \cdot 0 = 0
\end{aligned}$$

Finally, we see that $f(x) + 0 = f(a) + f(x) - f(a)$, we have

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} [f(a) + f(x) - f(a)], \\
&= \lim_{x\to a} [f(a)] + \lim_{x\to a} [f(x) - f(a)], \\
&= f(a) + 0 = f(a).
\end{aligned}$$

Therefore f is continuous and $f \in C([-1,1])$.
By definition, $C^{(1)}([-1,1]) \subseteq C([-1,1])$.

Lesson 2: Relations and Functions- Suggested Problems:

1. Convert each of the following decimal integers into binary representation

   A. Base 10 representation: 104
   B. Base 10 representation: 255
   C. Base 10 representation: 917
   D. Base 10 representation: 3
   E. Base 10 representation: $12,781$

2. Convert each of the following binary numbers into decimal representation

   A. Base 2 representation: 1001101
   B. Base 2 representation: 1101
   C. Base 2 representation: 1111111
   D. Base 2 representation: 101
   E. Base 2 representation: 1001101110101

3. Identify possible options for the domain space, domain, codomain and range for each of the following relations:

   A. $\{(x,y) \in \mathbb{R} \times \mathbb{R} \,:\, x = \sin(y)\}$
   B. $\left\{(x,y) \in \mathbb{R} \times \mathbb{R} \,:\, y = e^{-x^2}\right\}$
   C. $\{(x,y) \in \mathbb{R} \times \mathbb{R} \,:\, x^2 + y^2 = 25\}$
   D. $\left\{(x,y) \in \mathbb{R} \times \mathbb{R} \,:\, y = \dfrac{|x|}{x} \text{ for } x \neq 0\right\}$

4. Assuming the domain of each of the following function is the largest possible subset of $\mathbb{R}$, find the domain and range of:

   A. $f(x) = \dfrac{x^2 - 7x + 12}{x - 4}$
   B. $f(x) = \sqrt{15 - x}$
   C. $f(x) = \sqrt{x+4} + \sqrt{-4-x}$
   D. $f(x) = \left|\dfrac{x+4}{2}\right| + \left|\dfrac{x-5}{2}\right|$

5. For the lower-case and upper-case letters, decompose the USA ASCII Code Chart into a composition of two functions:

$$\begin{aligned} f &: U \to \mathbb{N}, \\ g &: \mathbb{N} \to B_7 \end{aligned}$$

   where B_7 is the set of $7-$bit binary integers.

6. Prove that $P_n([0,1]) \subseteq C^{(\infty)}([0,1])$.

7. Prove that $C^{(\infty)}([0,1]) - P_n([0,1]) \neq \emptyset$.

8. Generate examples of relations you use in daily life not included in the Lesson 2 notes. Identify the domain space and codomain for each relation. Determine if the relation is a function or not.

9. Generate examples of functions you use in daily life not included in the Lesson 2 notes. Identify the domain space and codomain for each function.