### 1.5 Sets and Operations on Sets

Most results in applied linear algebra depend on a formal rigorous logical structure known as set theory. In this section, we study sets and introduce a general framework to state the many concepts and propositions we will need in applied linear algebra.

In most lower-division calculus textbooks, functions are introduced using nonset theoretic terminology. In such books, functions are referred to as a mappings that have at most three independent variables and a single, real-valued output. The goal of first-year calculus is to introduce the formal mathematical theory used to analyze the behavior of functions using only a algebraic and graphical descriptions of the relationship between input and output variables. However, using set theory, we formalize the definition of a function. Instead of confining ourselves to maps whose range is contained in the real number line, we instead generalize functions as relationships between sets that satisfy important conditions. Before we do so, we need to build intuitions about sets.

A set is a collection of objects, known as elements of a set. To define a set of elements, we can use two different, but related, approaches:
i. the element enumeration method
ii. set-builder notation

When using the element enumeration method to define a set, we list all elements of the set, one by one. We use the left bracket symbol" $\{$ " to represent the start of our list of elements and the right bracket symbol "\}" to represent the end of our list. All objects that appear between these brackets are elements of our set. Commas represent the boundary between separate elements in our list. Uppercase letters in math font (slightly italicized) usually denote the names of sets.

## EXAMPLE 1.5.1

Lets define the set of lowercase letters in the English alphabet. We will call this set $L$ and define

$$
L=\{\mathrm{a}, \mathrm{~b}, \mathrm{c}, \mathrm{~d}, \mathrm{e}, \mathrm{f}, \mathrm{~g}, \mathrm{~h}, \mathrm{i}, \mathrm{j}, \mathrm{k}, \mathrm{l}, \mathrm{~m}, \mathrm{n}, \mathrm{o}, \mathrm{p}, \mathrm{q}, \mathrm{r}, \mathrm{~s}, \mathrm{t}, \mathrm{u}, \mathrm{v}, \mathrm{w}, \mathrm{x}, \mathrm{y}, \mathrm{z} .\}
$$

We can also define the set of uppercase letters in the English alphabet. We call this set $U$ and define
$U=\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}, \mathrm{G}, \mathrm{H}, \mathrm{I}, \mathrm{J}, \mathrm{K}, \mathrm{L}, \mathrm{M}, \mathrm{N}, \mathrm{O}, \mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{S}, \mathrm{T}, \mathrm{U}, \mathrm{V}, \mathrm{W}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}$.
By definition of our sets above, we see that the lowercase letter "d" is an element of $L$ while "d" is not an element of $U$. Similarly, the upper case letter "Z" in not in $L$ but is in $U$.

When we list all the elements of a set, we determine which objects are part of the set and which objects are not. The order in which the elements are listed does not matter, nor does repeated mention of a single element.

## EXAMPLE 1.5.2

Lets define the set of lowercase vowels in the English alphabet. We will call this set $V$ and define

$$
V=\{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}
$$

This set can also be written in a different order $V=\{\mathrm{u}, \mathrm{i}, \mathrm{o}, \mathrm{a}, \mathrm{e}\}$ or it can be listed with repeated elements $V=\{a, a, a, e, i, i, o, o, o, o, u, u\}$. Each of the representations of $V$ above define the same set, which is the set of lower case English vowels. From any of these descriptions of the set $V$, we see that the letter "e" is a lowercase vowel while the letter " $k$ " is not a lowercase vowel.

Often, we want to refer to individual elements of sets. To do so, we use the set membership symbol " $\in$ ". We write $x \in A$, read " x is an element of A " or " x is in A" to indicate that the element $x$ is contained in the set $A$. The expression $x \in A$ is a proposition, in the sense that it has a truth value. We say that $x \in A$ is true if and only if $x$ is an element of $A$. We say $x \in A$ is false if and only if $x$ is not an element of set $A$, written as $x \notin A$. The expression $x \notin A$ is the negation of the expression $x \in A$.

Sometimes we want to define sets using the element enumeration method without having to explicitly write every element in our set. In this case, we can use the ellipsis "..." to define our range of elements.

## EXAMPLE 1.5.3

Lets define the set of the first ten positive integers. We will call this set

$$
[10]=\{1,2,3, \ldots, 10\}=\{1,2,3,4,5,6,7,8,9,10\}
$$

The first description of this set suggests to the reader that we want to list all integers ranging from 1 to 10 while the second description lists out each of these. The benefit of using the ellipsis "..." is brevity and ease of reference. We can read the symbols "..." to mean "and so on." When defining the set [10], we can read our definition as " $[10]$ is the set of numbers including $1,2,3$, and so on until $10 . "$

When using the ellipsis with the element enumeration method, it is important to guard against ambiguity.

## EXAMPLE 1.5.4

Lets define the set:


This description is ambiguous. Perhaps we mean to define $E$ as the even numbers ranging from 2 to 16 , given by the set $E=\{2,4,6,8,10,12,14,16\}$. Or maybe we want to list the exponent values of 2 between 2 and 16 , defined by $E=\{2,4,8,16\}$. Formally speaking, our reader will not know which we intended without extra information. Often, we can make clear our use of the ellipsis for defining sets by adding context. We might say, let $E=\{2,4, \ldots, 16\}$ be the set of the first eight positive, even integers. On the other hand, we can say let $E=\{2,4, \ldots, 16\}$ be the set of the first four positive, integer powers of two. Each of these descriptions add context and thus allow for skillful use of the ellipsis notation for simplification.

We can also list elements of infinite sets using the ellipsis. This comes in very handy when we want to define a set with an infinite number of elements following a special, easily discernible pattern.

## EXAMPLE 1.5.5

Lets define the set of natural numbers, denoted as $\mathbb{N}$, to be the set of positive integers:

$$
\mathbb{N}=\{1,2,3,4, \ldots\}
$$

We intend this description to represent the set of positive integers, starting at one, incrementing by 1 and going on to infinity. We listed the first four elements of the set in order to assist the reader in discovering the pattern by which we might create future elements of this set. We see from this notation that to create the next element of our set from the last one, we add one. We might also assist in pattern discovery in other ways.

Any author who uses an ellipses should take responsibility to ensure that the surpressed pattern clearly understandable.

## EXAMPLE 1.5.6

Lets define the set of positive integers whose value is one less than a power of two. We will call this set $P$ defined as:

$$
P=\left\{1,3,7, \ldots, 2^{k}-1, \ldots\right\}
$$

Because there is some ambiguity about our intended pattern, we use the expression $2^{k}-1$ to specify a generic element of our set. This helps our reader get a sense of the desired pattern and add to the clarity of our exposition.

The second, and more effective, method of defining sets is known as set-builder notation. The general structure used in this paradigm is to list the set definition as $A=\{x: P(x)\}$, read using the following convention:


For a given $x$, the statement $P(x)$ should either be true or false. In the language of mathematical logic, $P(x)$ is a proposition with an associated truth value of true or false. If the statement $P(x)$ is true for $x$, then we say that $x$ is an element of our set. If $P(x)$ is false for a specific $x$, then $x$ is not in our set.

## EXAMPLE 1.5.7

Lets define the set of positive integers between 1 and 1024 . We will call this set $T$ defined as:

$$
T=\{x: x \text { is an integer and } 1 \leq x \leq 1024\}
$$

In this example, the proposition $P(x)$ is the sentence " $x$ is an integer and $1 \leq x \leq$ 1024." For any $x$, we can easily evaluate the truth value of $P(x)$. If $x=4$, we see $P(4)$ is true since 4 is an integer and $1 \leq 4 \leq 1024$ and therefor we know that $4 \in T$. On the other hand, for the integer $x=-1, P(-1)$ evaluates as false since -1 is outside our allowable range. We conclude $-1 \notin T$. Finally, if we want to check if the lowercase letter "b" is in $T$, we find the truth value of $P(\mathrm{~b})$. In this case $P(\mathrm{~b})$ is false since the letter "b" is not an integer. We conclude that $\mathrm{b} \notin T$.

There is some flexibility allowed when using set-builder notation. Some authors like to use the vertical bar "|" instead of the colon ":" to separate variable names from the defining proposition of the set:

$$
\{x \mid P(x)\}=\{x: P(x)\} .
$$

Both of these notations represent the same meaning. In this textbook, we will use the colon exclusively. Another variation in set-builder notation is the variable
name chosen to specify elements. The letter used to define the proposition $P(x)$ is inconsequential. We can write

$$
\{x: P(x)\}=\{y: P(y)\}=\{\theta: P(\theta)\}
$$

Depending on the context, wise authors chose variable names that are suggestive of the target application.

## EXAMPLE 1.5.8

Lets define the set angles between 0 and $2 \pi$. We will call this set:

$$
R_{[0,2 \pi]}=\{\theta: \theta \text { is an angle, measured in radians, and } 0 \leq \theta \leq 2 \pi\}
$$

Here, we choose the name of the set to be suggestive of a interval of radians and our variable name of $\theta$ follows a popular convention in mathematics to use the greek letter $\theta$ to name angles.

## Definition 1.1: Important Number Systems

There are a number of very popular sets that we will be using in this book. These sets show up in many fields in mathematics and have standard names. Below we define each of these for future reference:
$\mathbb{N}=\{1,2,3,4,5, \ldots\}$, the set of natural numbers or positive integers.
$\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3, \ldots\}$, the set of integers.
$\mathbb{Q}=\{x: x=p / q$ for some $p, q \in \mathbb{Z}, q \neq 0\}$, the set of rational numbers.
$\mathbb{R}=\{x: x$ is a real number $\}$, the set of real numbers.
$\mathbb{C}=\{x: x=a+b i$ for some $a, b \in \mathbb{R}, i=\sqrt{-1}\}$, the set of complex numbers.

Each of the above number systems can be constructed using a finite list of axioms. Generally, these axioms can categorized into algebraic axioms or ordering axioms. The algebraic axioms of any of the number systems above indicate important concepts of equality. On the other hand, the order axioms give relationships based on the order between any two elements.

It is a nontrivial mathematical problem to construct a complete mathematical system to describe the real-number system. The study of this problem and related issues is called Real Analysis. In this textbook, we assume familiarity with major results in real analysis and calculus. Specifically, we assume familiarity with both the algebraic and order axioms of the real numbers and knowledge of the existence of important results on fundamental inequalities (such as Cauchy's inequality, Minkowski's inequality, and Holder's inequality). We assume familiarity with limits, derivatives, integrals and sequences and series. All of these concepts follow from the study of the number systems listed above.

## EXAMPLE 1.5.9

Let $a, b \in \mathbb{R}$ with $a<b$. We define

$$
\begin{aligned}
(a, b) & =\{x \in \mathbb{R}: a<x<b\} \\
{[a, b] } & =\{x \in \mathbb{R}: a \leq x \leq b\} \\
{[a, b) } & =\{x \in \mathbb{R}: a \leq x<b\} \\
(a, b] & =\{x \in \mathbb{R}: a<x \leq b\} \\
(-\infty, a) & =\{x \in \mathbb{R}: x<a\} \\
(-\infty, a] & =\{x \in \mathbb{R}: x \leq a\} \\
(a, \infty) & =\{x \in \mathbb{R}: a<x\} \\
{[a, \infty) } & =\{x \in \mathbb{R}: a \leq x\} \\
(-\infty, \infty) & =\mathbb{R}
\end{aligned}
$$

We call any interval in the form $(\alpha, \beta)$ an open interval while we sat that intervals in the form $[\alpha, \beta]$ are closed intervals. Intervals $[\alpha, \beta)$ and $(\alpha, \beta]$ are half-open intervals. In this case, $\alpha$ and $\beta$ are possibly $\pm \infty$

## Definition 1.2: Subset

Given two sets $A$ and $B$, we say that $A$ is a subset of $B$, written $A \subseteq B$, if and only if every element of $A$ is also an element of $B$. If there is an element of $A$ not contained in $B$, then we say $A$ is not a subset of $B$ and we write $A \nsubseteq B$.

## EXAMPLE 1.5.10

Recall the set of lowercase vowels $V$ from Example 1.5.1 as well as the set of lowercase letters $L$ and the set of uppercase letters $U$ from Example 1.5.2. We see that $V$ is a subset of $L$, written

$$
V \subseteq L
$$

However, since a $\in V$ and a $\notin U, V$ is not a subset of the set of uppercase letters $U$. We write this as

## EXAMPLE 1.5.11

Any interval is a subset of $\mathbb{R}$. In particular, we see

$$
(-2,4) \subseteq \mathbb{R}
$$

In order to show that a set $A$ is a subset of another set $B$, we need to show that every element of $A$ is also an element of $B$. A direct proof of this claim follows a very specific format, given below.

## Direct Proof of $A \subseteq B$ <br> Proof:

Suppose $x \in A$.

Therefore, $x \in B$.
By definition, $A \subseteq B$.

To establish this inclusion, the properties of set $A$ and $B$ will often come into play. The vertical ellipsis included in the proof structure above represent work done by the author of a proof to establish that every element of $A$ satisfies the conditions for inclusion in the set $B$.

## EXAMPLE 1.5.12

Let $A=\{\pi / 4,9 \pi / 4\}$. Let $B=\{\theta \in[0,6 \pi]: 4 \cos (\theta-\pi / 4)-4=0\}$. We want to show that $A \subseteq B$. In the proof below, we will establish our subset relation by checking each element of $A$ individually. For small sets, this option may be feasible. Proof: Suppose that $\theta \in A$. Then $\theta=\pi / 4$ or $\theta=9 \pi / 4$. If $\theta=\pi / 4$, we know

$$
4 \cos (\pi / 4-\pi / 4)-4=4-4=0
$$

For $\theta=9 \pi / 4$, we see

$$
4 \cos (9 \pi / 4-\pi / 4)-4=4-4=0
$$

In each case, $\theta \in B$. By definition, we see that $A \subseteq B$.
The set $A$ is a proper subset of the set $B$ if and only if $A \subseteq B$ and there exists an element $x \in B$ such that $x \notin A$. Some authors write the proper subset relation as $A \subset B$ or $A \subsetneq B$. In this case, we may say that $A$ is strictly contained in $B$.

## EXAMPLE 1.5.13

The subset relations between the most popular numbers systems are as follows

$$
\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}
$$

Each of the inclusions above are strict inclusions, meaning that each subset is a strict subset of each larger set. While all elements of $\mathbb{N}$ are also in $\mathbb{Z}$, the reverse relation is not true. Specifically, $-1 \in \mathbb{Z}$ and $-1 \notin \mathbb{N}$. Similarly, we see that $1 / 2 \in \mathbb{Q}$ and $1 / 2 \notin \mathbb{Z}$. To prove that $\sqrt{2} \in \mathbb{R}$ and $\sqrt{2} \notin \mathbb{Q}$ is a fun mathematical exercise assigned as a challenge problem. Finally, since $\sqrt{-1} \in \mathbb{C}$ while $\sqrt{-1} \notin \mathbb{R}$, we see that $\mathbb{R}$ is a proper subset of $\mathbb{C}$.

Given sets $A$ and $B$, we say these sets are equal if an only if they contain exactly identical elements. To verify that $A=B$, we need to check that $x \in A$ if and only if $x \in B$. We can use subsets to establish this equivalence relation:

$$
A=B \text { if and only if } A \subseteq B \text { and } B \subseteq A
$$

We can often prove that two sets are equal using a direct proof.

## Direct Proof of $A=B$

Proof:
i. Prove $A \subseteq B$
ii. Prove $B \subseteq A$
iii. Conclude $A=B$.

## EXAMPLE 1.5.14

Let $A=\left\{x \in \mathbb{R}: \frac{d}{d x}\left[x^{3}-12 x\right]=0\right\}$ and $B=\{-2,2\}$. Then $A=B$.
Proof: We want to show that $A=B$. We will do so by direct proof.
i. To establish that $A \subseteq B$, let $x \in A$. By taking the derivative of $x^{3}-12 x$ and setting it equal to zero, we know that

$$
3 x^{2}-12=0
$$

Factoring our derivative polynomial yields $3(x-2)(x+2)=0$. Thus $x=-2$ or $x=2$ and $x \in B$. Therefore $A \subseteq B$.
ii. To show $B \subseteq A$, we check each element individually. By substituting each element of $B$ into our equation $\frac{d}{d x}\left[x^{3}-12 x\right]=0$, we see that -2 and 2 are solutions to $3 x^{2}-12=0$. Therefore $B \subseteq A$.
iii. By (i.) and (ii.) above, $A=B$.

An important special subset is known as the empty set.

## Definition 1.3: Empty Set

Let $\emptyset=\{x: x \neq x\}$. This set is called the empty set since it has no elements.

We assume, via an axiom, that $\emptyset$ exists. Since there are no elements in $\emptyset$, the statement $x \in \emptyset$ is false for all objects $x$.

## Theorem 1: Classic Subset Results

Let $A, B$ and $C$ be any sets. Then
a. $\emptyset \subseteq A$.
b. $A \subseteq A$.
c. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof. The proofs of these statements are good challenge problems. To begin these proofs, look back at the method for direct proof of $A \subseteq B$. You will need to learn about propositions and formal mathematical logic (including truth tables and conditional statements) in order to fully understand these proofs).

## Lesson 1: Suggested Problems

1. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
2. Prove that $X=Y$ where $X=\left\{x \in \mathbb{Z},: x^{2}<10\right\}$ and $Y=\{0,1,-1,2,-2,3,-3\}$
3. List all proper subsets of each of the following sets
a. $\emptyset$
b. $[3]=\{1,2,3\}$
c. $\{a, b\}$
4. Give an example of sets $A, B, C$ such that each of the following is true. If no such sets exists, indicate that the relationship is not possible by writing "impossible."
a. $A \subseteq B, B \nsubseteq C$ and $A \subseteq C$
b. $A \subseteq B, B \subseteq C$ and $C \subseteq A$
c. $A \nsubseteq B, B \nsubseteq C$ and $A \subseteq C$

## Lesson 1: Challenge Problems

1. Prove that $\sqrt{2} \notin \mathbb{Q}$
2. For any natural number $c$, define the set

$$
c \mathbb{Z}=\{z \in \mathbb{Z}: z=c \cdot n \text { for } n \in \mathbb{Z}\}
$$

In other words, let $c \mathbb{Z}$ be the set of all integer multiples of $c$. Prove that for any $n, m \in \mathbb{N}, n=m$ if and only if $n \mathbb{Z}=m \mathbb{Z}$.

