## Math 2B: Applied Linear Algebra

True/False For the problems below, circle $T$ if the answer is true and circle $F$ is the answer is false.

1. (T) F There are three types of elementary row reductions we will use to solve linear systems.
2. T F Let $A, \mathbf{b}$ be given. If $f(\mathbf{x})=A \mathbf{x}$, then a solution to the corresponding set of linear system $A \mathbf{x}=\mathbf{b}$ exists if and only if $\mathbf{b}$ is in the codomain of $f$
3. T F The homogeneous equation $A \mathbf{x}=\mathbf{0}$ can be inconsistent.
4. (T) F If $A \in \mathbb{R}^{m \times n}$ such that $A \mathbf{x}=\mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^{m}$, then $A$ has $m$ pivot columns.
5. (T) F If a system of linear equations has two different solutions, it must have infinitely many solutions.
6. T F Let $A \in \mathbb{R}^{m \times n}$ and suppose $\mathbf{x} \in \mathbb{R}^{n}$. Suppose $\mathbf{b} \in \mathbb{R}^{m}$ is nonzero. Suppose $\mathbf{x}_{1}^{*}$ and $\mathbf{x}_{2}^{*}$ are solutions to the inhomogeneous system $A \mathbf{x}=\mathbf{b}$. Then any linear combination $c_{1} \mathbf{x}_{1}^{*}+c_{2} \mathbf{x}_{2}^{*}$ is a solution to the linear system $A \mathbf{x}=\mathbf{b}$.
7. (T) F Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^{m}$ be given. The solution set of an inhomogeneous equation $A \mathbf{x}=\mathbf{b}$ is the set of all vectors of the form $\mathbf{x}=\mathbf{x}^{*}+\mathbf{z}$ where $\mathbf{x}^{*}$ is a particular solution to the linear system and $\mathbf{z}$ is a solution to the homogeneous system $A \mathbf{x}=\mathbf{0}$.
8. (T) F If a system $A \mathbf{x}=\mathbf{b}$ has more than one solution, then so does the system $A \mathbf{x}=\mathbf{0}$.
9. T F If a system of linear equations has no free variables, then is always a unique solution.
10. T F The superposition principle for inhomogeneous systems generalizes the invertible matrix theorem by classifying completely the existence and uniqueness results for a general linear system $A \mathbf{x}=\mathbf{b}$, where $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^{m}$ are given, and $\mathbf{x} \in \mathbb{R}^{n}$ is unknown.
11. T F The homogeneous equation $A \mathbf{x}=\mathbf{0}$ has nonzero solution if and only if the equivalent reduced row echelon form of the matrix equation has at least one free variable.
12. $\mathbf{T}$ F Given $A$ and $\mathbf{b}$ of appropriate dimensions, the linear system $A \mathbf{x}=\mathbf{b}$ has a solution if and only if $\mathbf{b}$ can be written as a linear combination of the vectors $\{A(:, 1), A($ : $, 2), \ldots, A(:, n)\}$.
13. T F If $A \in \mathbb{R}^{m \times n}$ has $m$ linearly independent rows, then the solution to $A \mathbf{x}=\mathbf{b}$ is unique for each $\mathbf{b} \in \mathbb{R}^{m}$.
14. (T) F Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^{m}$ be given. The solution set to a linear system with an augmented matrix

$$
\left[\begin{array}{ll}
A & \mathbf{b}
\end{array}\right]
$$

is identical to the solution set to the linear systems problem $A \mathbf{x}=\mathbf{b}$.
15. T F If $A \in \mathbb{R}^{n \times n}$ has $n$ linearly independent columns, then $A$ is row equivalent to the $n \times n$ identity matrix $I_{n}$.
16. T F If $A \in \mathbb{R}^{m \times n}$ has $m$ linearly independent rows, then the function $f(\mathbf{x})=A \mathbf{x}$ is a one-to-one mapping.
17. T F The general solution to a linear system problem is a description of every element of the solution set to the given linear system.
18. T F Let $A, \mathbf{b}$ be given. If $f(\mathbf{x})=A \mathbf{x}$, then a solution to the corresponding set of linear system $A \mathbf{x}=\mathbf{b}$ exists if and only if $\mathbf{b}$ is in the range of $f$
19. T F Let $A, \mathbf{x}$ be given. If $g(\mathbf{b})=A^{T} \mathbf{b}$, then a solution to the corresponding set of linear system $A^{T} \mathbf{b}=\mathbf{x}$ exists if and only if $\mathbf{x}$ is in the range of $g$
20. T F Suppose we have the basis $\left\{\mathbf{z}_{1}, \ldots, \mathbf{z}_{d}\right\}$ for $\operatorname{Nul}(A)$ with $d \leq n$. Suppose we have $\mathbf{x}_{1}^{*}$ as a particular solution to a given inhomogeneous system $A \mathbf{x}=\mathbf{b}$. Then we can construct any solution to this system as $\mathbf{x}=\mathbf{x}_{1}^{*}+\sum_{j=1}^{d} c_{j} \mathbf{z}_{j}$.
21. T F Consider the linear system with $m$ equations and $n$ unknowns given by

$$
\begin{gathered}
a_{11} x_{1}+a_{12} x_{2}+\cdots+a_{1 n} x_{2}=b_{1} \\
a_{21} x_{1}+a_{22} x_{2}+\cdots+a_{2 n} x_{2}=b_{2} \\
\vdots \\
a_{m 1} x_{1}+a_{m 2} x_{2}+\cdots+a_{m n} x_{2}=b_{m}
\end{gathered}
$$

where scalars $a_{i k}, b_{i} \in \mathbb{R}$ are given for all $i \in\{1,2, \ldots, m\}$ and $k \in\{1,2, \ldots, n\}$ and variables $x_{1}, x_{2}, \ldots, x_{n}$ are unknown real numbers. The solution set of this linear system is a list of $n$ numbers $y_{1}, y_{2}, \ldots, y_{n}$ such that if each $y_{k}$ is substituted for the corresponding $x_{k}$, then all $m$ equations will be true.
22. T F Any two linear systems problems with the same solution set must be equivalent linear systems.
23. T F To solve a linear system, it is sufficient to find a general parametric equation that describes all elements in the solution set of this linear system.
24. T F All linear systems that have a coefficient matrix with free variables have infinitely many solutions.
25. T F Let $A, \mathbf{b}$ be given. If $f(\mathbf{x})=A \mathbf{x}$, then a solution to the corresponding set of linear system $A \mathbf{x}=\mathbf{b}$ exists if and only if $\mathbf{b}$ is in the codomain of $f$
26. T F If $f(\mathbf{x})=A \mathbf{x}$ and $\mathbf{b} \in \operatorname{Rng}(f)$, then $\mathbf{b}$ is linearly independent from the columns of $A$.
27. (T) F Let $A, \mathbf{x}$ be given. If $g(\mathbf{b})=A^{T} \mathbf{b}$, then a solution to the corresponding set of linear system $A^{T} \mathbf{b}=\mathbf{x}$ exists if and only if $\mathbf{x}$ is in the range of $g$
28. T F If $\mathbf{b} \notin \operatorname{Col}(A)$, then we want to find a solution to $A \mathbf{x}=\mathbf{b}$ using $\operatorname{RREF}(A)$ and the superposition principle.
29. T F The superposition principle for inhomogeneous systems generalizes the invertible matrix theorem by classifying completely the existence and uniqueness results for a general linear system $A \mathbf{x}=\mathbf{b}$, where $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^{m}$ are given, and $\mathbf{x} \in \mathbb{R}^{n}$ is unknown.
30. T F Solving linear systems using elementary row reductions is equivalent to changing a matrix equation using linear combinations on the rows of that matrix.
31. T F If $A \in \mathbb{R}^{m \times n}$ such that $A \mathbf{x}=\mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^{m}$, then $A$ has $m$ pivot columns.
32. T F Consider the linear systems problem

$$
A \mathbf{x}=\mathbf{b}
$$

where matrix $A \in \mathbb{R}^{m \times n}$ and vector $\mathbf{b} \in \mathbb{R}^{m}$ are given and vector $\mathbf{x} \in \mathbb{R}^{n}$ is unknown and desired. If this linear system is inconsistent, there may be an $\mathbf{x} \in \mathbb{R}^{n}$ such that

$$
\|\mathbf{b}-A \mathbf{x}\|_{2}=0
$$

Multiple Choice For the problems below, circle the correct response for each question.

1. Let

$$
A=\left[\begin{array}{rrrr}
1 & -2 & 0 & 3 \\
2 & -3 & -1 & -4 \\
3 & -5 & -1 & -1
\end{array}\right], \quad \mathbf{b}=\left[\begin{array}{l}
1 \\
1 \\
2
\end{array}\right] .
$$

By performing elementary row operations on this matrix we see that $A$ is row equivalent to the matrix

$$
U=\left[\begin{array}{rrrr}
1 & -2 & 0 & 3 \\
0 & 1 & -1 & -10 \\
0 & 0 & 0 & 0
\end{array}\right] .
$$

Which of the following represents the solution set $S$ of the linear system $A \mathbf{x}=\mathbf{b}$ ?
A. $S=\left\{\mathbf{x} \in \mathbb{R}^{4}: \mathbf{x}=\left[\begin{array}{l}1 \\ 0 \\ 1 \\ 0\end{array}\right]+c_{1}\left[\begin{array}{r}1 \\ -1 \\ 1 \\ 0\end{array}\right]+c_{2}\left[\begin{array}{r}-3 \\ 10 \\ 0 \\ 1\end{array}\right]\right\}$
B. $S=\left\{\mathbf{x} \in \mathbb{R}^{4}: \mathbf{x}=\left[\begin{array}{l}1 \\ 0 \\ 1 \\ 0\end{array}\right]+c_{1}\left[\begin{array}{l}2 \\ 1 \\ 1 \\ 0\end{array}\right]+c_{2}\left[\begin{array}{r}-17 \\ -10 \\ 0 \\ 1\end{array}\right]\right\}$
C. $S=\left\{\mathbf{x} \in \mathbb{R}^{4}: \mathbf{x}=\left[\begin{array}{l}1 \\ 0 \\ 1 \\ 0\end{array}\right]+c_{1}\left[\begin{array}{l}2 \\ 1 \\ 1 \\ 0\end{array}\right]+c_{2}\left[\begin{array}{r}17 \\ 10 \\ 0 \\ 1\end{array}\right]\right\}$
D. $A \mathbf{x} \neq \mathbf{b}$ for all $\mathbf{x} \in \mathbb{R}^{4}$.
E. None of these
2. Suppose $A \in \mathbb{R}^{m \times n}$. Given a nonzero vector $\mathbf{b} \in \mathbb{R}^{m}$, suppose that you know:
I. Vectors $\mathbf{z}_{1}, \mathbf{z}_{2} \in \mathbb{R}^{n}$ solve the linear system problem $A \mathbf{x}=\mathbf{0}$
II. Vectors $\mathbf{x}^{*}, \mathbf{y}^{*} \in \mathbb{R}^{n}$ solve the linear system problem $A \mathbf{x}=\mathbf{b}$.

Which of the following is NOT a solution for the linear system problem $A \mathbf{x}=\mathbf{b}$ ?
A. $\mathrm{x}^{*}+\mathrm{z}_{1}$
B. $\mathbf{y}^{*}+\mathrm{z}_{2}$
C. $\mathrm{x}^{*}+\mathrm{y}^{*}$
D. $3 \mathbf{z}_{1}+\mathbf{x}^{*}-4 \mathbf{z}_{2}$
E. $2 \mathbf{x}^{*}-\mathbf{y}^{*}$

## Free Response

1. Consider the following general linear-systems problem:

A. How many linearly independent solutions to $A \cdot \mathbf{x}=\mathbf{0}$ are there? Find all linearly independent solutions to this homogeneous equation $A \cdot \mathbf{x}=\mathbf{0}$ ?

Solution: Let's begin by transforming our original system into an equivalent system with a coefficient matrix in RREF. To do so, we use our calculator and create the system

$$
\underbrace{\left[\begin{array}{rrrrrrrr}
1 & 0 & -4 & 1 & 0 & 2 & 0 & 4 \\
0 & 1 & 2 & 3 & 0 & -1 & 0 & -3 \\
0 & 0 & 0 & 0 & 1 & 3 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -3
\end{array}\right]}_{U} \cdot \underbrace{\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4} \\
x_{5} \\
x_{6} \\
x_{7} \\
x_{8}
\end{array}\right]}_{\mathbf{x}}=\underbrace{\left[\begin{array}{l}
1 \\
2 \\
1 \\
1
\end{array}\right]}_{\mathbf{y}}
$$

Now recall the following theorem:
Theorem 1: The Solution Set for $A \cdot \mathrm{x}=0$

Suppose $A \in \mathbb{R}^{m \times n}$ is given. Suppose that $A$ has a total of $0 \leq d \leq n$ nonpivot columns. Let $\mathbf{z}_{1}, \mathbf{z}_{2}, \ldots, \mathbf{z}_{d} \in \mathbb{R}^{n}$ be linearly independent solutions to the homogeneous linear-systems problem $A \cdot \mathbf{x}=\mathbf{0}$. If $\mathbf{z} \in \mathbb{R}^{n}$ is any solution of the homogeneous linearsystems problem $A \cdot \mathbf{x}=\mathbf{0}$, then we can write $\mathbf{z}$ as a linear combination of the vectors $\mathbf{z}_{1}, \mathbf{z}_{2}, \ldots, \mathbf{z}_{d}$.

Thus, we know that the number of linearly independent solutions to the system $A \mathbf{x}=\mathbf{0}$ is equal to the number of nonpivot columns of $A$. We also know that the number of nonpivot columns of $A$ is equal to the number of nonpivot columns of $U=\operatorname{RREF}(A)$. Using our matrix $U$ from above, we see $A$ has 4 pivot columns. Then, we expect a total of $d=8-4=4$ nonpivot columns and thus four linearly independent solutions to the homogeneous linear-systems problem.

Solution: Now, we attempt to find the all 4 linearly independent solution (one for each nonpivot column). Notice that the nonpivot columns of $U$ are columns 3, 4, 6 and 8 . Moreover, each of the nonpivot column of $U$ can be written as a linear combination of the pivot columns. Indeed, one of the most powerful features of the RREF is our ability to quickly ascertain the linear dependence relations between the columns of this matrix.

Nonpivot column 1: Let's consider column 3, our first nonpivot column. We can write:

$$
U(:, 3)=\left[\begin{array}{r}
-4 \\
2 \\
0 \\
0
\end{array}\right]=-4 \cdot\left[\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right]+2 \cdot\left[\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right]=-4 \cdot U(:, 1)+2 \cdot U(:, 2)
$$

In other words, we can find a linear combination of columns 1 and 2 with nonzero coefficients that sum to zero, in the form

$$
4 \cdot U(:, 1)+-2 \cdot U(:, 2)+1 \cdot U(:, 3)=4 \cdot\left[\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right]+-2 \cdot\left[\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right]+1 \cdot\left[\begin{array}{r}
-4 \\
2 \\
0 \\
0
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0 \\
0
\end{array}\right]=\mathbf{0}
$$

Using this equation, we can construct vector $\mathbf{z}_{1} \in \mathbb{R}^{8}$ to encodes the linear dependence relationships in the columns of $U$ as a solution to the homogeneous linear-systems problem:

$$
U \cdot \mathbf{z}_{1}=\left[\begin{array}{rrrrrrrr}
1 & 0 & -4 & 1 & 0 & 2 & 0 & 4 \\
0 & 1 & 2 & 3 & 0 & -1 & 0 & -3 \\
0 & 0 & 0 & 0 & 1 & 3 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -3
\end{array}\right]\left[\begin{array}{r}
4 \\
-2 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right], \quad \text { with } \mathbf{z}_{1}=\left[\begin{array}{r}
4 \\
-2 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right]
$$

Continuing in this manner, we produce all four linearly independent solutions:

$$
\mathbf{z}_{1}=\left[\begin{array}{r}
4 \\
-2 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right], \quad \mathbf{z}_{2}=\left[\begin{array}{r}
-1 \\
-3 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{array}\right], \quad \mathbf{z}_{3}=\left[\begin{array}{r}
-2 \\
1 \\
0 \\
0 \\
-3 \\
1 \\
0 \\
0
\end{array}\right], \quad \mathbf{z}_{4}=\left[\begin{array}{r}
-4 \\
3 \\
0 \\
0 \\
-5 \\
0 \\
3 \\
1
\end{array}\right] .
$$

B. Find the solution set for this GLSP. Is the solution set to the GLSP a subspace of $\mathbb{R}^{8}$ ? (Hint: recall the definition of a subspace of a vector space)

Solution: We recall that any solution to our original linear-systems problem takes the form

$$
\mathrm{x}=\mathrm{x}^{*}+\mathrm{z}
$$

where $\mathbf{x}^{*}$ is a particular solution and $\mathbf{z}$ is any solution to the homogeneous linear system. In this case, we can write any solution as

$$
\mathbf{x}=\left[\begin{array}{l}
1 \\
2 \\
0 \\
0 \\
1 \\
0 \\
1 \\
0
\end{array}\right]+c_{1} \cdot\left[\begin{array}{r}
4 \\
-2 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right]+c_{2} \cdot\left[\begin{array}{r}
-1 \\
-3 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{array}\right]+c_{3} \cdot\left[\begin{array}{r}
-2 \\
1 \\
0 \\
0 \\
-3 \\
1 \\
0 \\
0
\end{array}\right]+c_{4} \cdot\left[\begin{array}{r}
-4 \\
3 \\
0 \\
0 \\
-5 \\
0 \\
3 \\
1
\end{array}\right] .
$$

