### 9.2 Solution Sets for General Linear Systems

We begin our discussion of solution sets by considering a toy general linearsystems probelm

$$\underbrace{\begin{bmatrix} -3 & 6 & -1 & 1 & -7\\ 1 & -2 & 2 & 3 & -1\\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -13\\ 6\\ 13 \end{bmatrix}}_{\mathbf{b}}$$

By multiplying our entire system on the left by a sequence of elementary matrices (or by using our calculator), we see that the equivalent system involving the RREF of A is given by

$$\underbrace{\begin{bmatrix} 1 & -2 & 0 & -1 & 3\\ 0 & 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U} \cdot \underbrace{\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 4\\ 1\\ 0 \end{bmatrix}}_{\mathbf{y}}$$

We can make a number of interesting observations about the matrix U. For example, we can immediately identify the pivot columns of U to be columns 1 and 3.

$$U = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Accordingly, the nonpivot columns of U are columns 2, 4 and 5. We also notice that each of the nonpivot columns of U can be written as a linear combination of the pivot columns. Indeed, one of the most powerful features of the RREF is our ability to quickly ascertain the linear dependence relations between the columns of this matrix. For example, let's consider column 2, our first nonpivot column. We can write:

$$U(:,2) = \begin{bmatrix} -2\\0\\0 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1\\0\\0 \end{bmatrix} = -2 \cdot U(:,1)$$

In other words, we can find a linear combination of columns 1 and 2 with nonzero coefficients that sum to zero, in the form

$$2 \cdot U(:,1) + U(:,2) = 2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

Using this equation, we can construct vector  $\mathbf{z}_1 \in \mathbb{R}^5$  that encodes the linear dependence relationships in the columns of U.

$$U \cdot \mathbf{z}_{1} = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{with } \mathbf{z}_{1} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

vS20190403

We can move onto the next nonpivot column:

$$U(:,4) = \begin{bmatrix} -1\\2\\0 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1\\0\\0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0\\1\\0 \end{bmatrix} = -1 \cdot U(:,1) + 2 \cdot U(:,3)$$

Again, we use vector arithmetic to set the right-hand side of this equation to zero and create the equivalent equation

$$1 \cdot U(:,1) - 2 \cdot U(:,3) + U(:,4) = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}.$$

This gives rise to a vector  $\mathbf{z}_2 \in \mathbb{R}^5$  such that

$$U \cdot \mathbf{z}_{2} = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{with } \mathbf{z}_{2} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Finally, we see that our last nonpivot column can be rewritten as

$$U(:,5) = \begin{bmatrix} 3\\-2\\0 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1\\0\\0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0\\1\\0 \end{bmatrix} = 3 \cdot U(:,1) - 2 \cdot U(:,3)$$

We set the right-hand side of this equation to zero to find

$$-3 \cdot U(:,1) + 2 \cdot U(:,3) + 1 \cdot U(:,5) = -3 \cdot \begin{bmatrix} 1\\0\\0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0\\1\\0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3\\-2\\0 \end{bmatrix} = \begin{bmatrix} 0\\0\\0 \end{bmatrix} = 0.$$

We create vector  $\mathbf{z}_3 \in \mathbb{R}^5$  such that

$$U \cdot \mathbf{z}_{2} = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{with } \mathbf{z}_{3} = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Using this information, we immediately list three linearly independent solutions to the linear-systems problem

$$U \cdot \mathbf{x} = \mathbf{0}$$

given by  $\mathbf{z}_1, \mathbf{z}_2$ , and  $\mathbf{z}_3$ . Moreover, we see that a particular solution for this system of equations is given by

$$\mathbf{x}^* = \begin{bmatrix} 4\\0\\1\\0\\0\end{bmatrix}.$$

The solution set the original linear system is given by

$$\mathbf{x}^* + c_1 \cdot \mathbf{z}_1 + c_2 \cdot \mathbf{z}_2 + c_3 \cdot \mathbf{z}_3$$

## **Definition 9.6: Homogeneous Linear System**

If  $A \in \mathbb{R}^{m \times n}$ , then the corresponding homogeneous linear-systems problem is to find all unknown vectors  $\mathbf{x} \in \mathbb{R}^n$  such that

 $A \cdot \mathbf{x} = \mathbf{0}.$ 

The homogeneous linear-systems problem is a special case of the general linearsystems problem where the  $m \times 1$  vector **b** on the right-hand side is zero. It is worth noting that homogeneous linear-systems problems always have at least one solution. In particular, for  $\mathbf{z} = \mathbf{0} \in \mathbb{R}^n$  we see

$$A \cdot \mathbf{z} = \begin{bmatrix} A(:,1) & A(:,2) & \cdots & A(:,n) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sum_{k=1}^{n} 0 \cdot A(:,k) = \mathbf{0} \in \mathbb{R}^{m}$$

To find any nonzero solution to a homogeneous linear systems problem, we look for linearly dependent columns of the matrix A. Such columns are most easily identified by transforming matrix A into RREF.

Theorem 32: Solution to  $A \cdot \mathbf{x} = \mathbf{0}$  using RREF

Let  $A \in \mathbb{R}^{m \times n}$  is a given matrix and suppose U = RREF(A). Then, for any  $\mathbf{x} \in \mathbb{R}^n$  we have

 $U \cdot \mathbf{x} = \mathbf{0}.$ 

 $A \cdot \mathbf{x} = \mathbf{0}$ 

*Proof.* Let  $A \in \mathbb{R}^{m \times n}$ . Suppose that U = RREF(A). In order to prove this biconditional theorem, we need to show:

- i. If  $A \cdot \mathbf{x} = \mathbf{0}$ , then  $U \cdot \mathbf{x} = \mathbf{0}$
- ii. If  $U \cdot \mathbf{x} = \mathbf{0}$ , then  $A \cdot \mathbf{x} = \mathbf{0}$

Let's begin by reviewing our algorithm for creating RREF(A). By construction of U, there exists a sequence of elementary matrices  $E_1, E_2, ..., E_t \in \mathbb{R}^{m \times m}$  such that

$$E_t \cdot E_{t-1} \cdots E_2 \cdot E_1 \cdot A = U$$

for some  $t \in \mathbb{N}$ . Moreover, each elementary matrix  $E_j$  can either be written as a shear matrix, dilation matrix, or permutation matrix, for  $1 \leq j \leq t$ . Because each of these matrices is nonsingular, then the  $m \times m$  matrix

$$E = E_t \cdot E_{t-1} \cdots E_2 \cdot E_1$$

is nonsingular. This follows from the fact that the any product of nonsingular matrices is nonsingular.

With this in mind, let's start with part (i.) of this proof which is the forward direction of our bi-conditional statement. In particular,

$A \cdot \mathbf{x} = 0$	$\Rightarrow$	$E \cdot A \cdot \mathbf{x} = E \cdot 0$
	$\Rightarrow$	$U \cdot \mathbf{x} = 0$

The final line follows since for  $E \cdot \mathbf{0} = \mathbf{0}$  for any  $E \in \mathbb{R}^{m \times m}$ .

Next, let's move to part (ii.) which is to show the reverse.

$$U \cdot \mathbf{x} = \mathbf{0} \qquad \implies \qquad (E \cdot A) \cdot \mathbf{x} = E \cdot \mathbf{0}$$
$$\implies \qquad E \cdot (A \cdot \mathbf{x}) = \mathbf{0}$$
$$\implies \qquad A \cdot \mathbf{x} = \mathbf{0}$$

The final line results from the fact that if  $E \in \mathbb{R}^{m \times m}$  is nonsingular, then we know  $E \cdot \mathbf{y} = \mathbf{0}$  if and only if  $\mathbf{y} = \mathbf{0}$ . in this case, we set  $\mathbf{y} = A \cdot \mathbf{x}$ . With this, we have finish our proof since both directions of this statement hold true.

The theorem above says that if U = RREF(A), then the solution sets of  $A \cdot \mathbf{x} = \mathbf{0}$ and  $U \cdot \mathbf{x} = \mathbf{0}$  are identical. In other words, to solve any homogeneous linearsystems problem, we can transform the coefficient matrix A into RREF and solve the equivalent system.

EXAMPLE 9.2.1

Recall general linear-systems problem from Example 9.1.2 to create a model of the acceleration of a Tesla model S.

$$\begin{bmatrix} 1 & 0.5 & 0.25 \\ 1 & 2.5 & 6.25 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 40 \end{bmatrix}$$

In that example, we used left multiplication by elementary matrices to transform our coefficient matrix A into coefficient matrix U = RREF(A) and stated the equivalent system

$$\begin{bmatrix} 1 & 0 & -1.25 \\ 0 & 1 & 3.00 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 15 \end{bmatrix}$$

Let's use our matrix U in RREF, where

$$U = \begin{bmatrix} 1 & 0 & -1.25 \\ 0 & 1 & 3.00 \end{bmatrix} = \text{RREF}(A)$$

to find nonzero solutions to the homogeneous linear

Theorem 33: Solution to  $A \cdot \mathbf{x} = \mathbf{0}$ 

Suppose  $A \in \mathbb{R}^{m \times n}$  is a given matrix. Then, the number of linearly independent solutions to the homogeneous linear-systems problem

 $A \cdot \mathbf{x} = \mathbf{0}$ 

is equal to the number of nonpivot columns of A.

The theorem above is very helpful from the standpoint of theory. To find the number of nonpivot columns of A, recall that

- i. p = # of pivot columns of U, where U = RREF(A) is the reduced row echelon of A.
- ii. p is the number of linearly independent columns of A.
- iii. the number of nonpivot columns of A equals d = (n p)
- iv. the number of nonpivot columns of A is the number of linearly dependent columns of A
- v. d is the number of linearly independent solutions to the homogeneous solution  $A \cdot \mathbf{x} = \mathbf{0}$

# Theorem 34: Superposition of Solutions for $A \cdot \mathbf{x} = \mathbf{0}$

Suppose  $A \in \mathbb{R}^{m \times n}$  is given. Suppose that A has a total of  $0 \leq d \leq n$  nonpivot columns. Let  $\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_d$  be linearly independent solutions to the homogeneous linear equation  $A \cdot \mathbf{x} = \mathbf{0}$ . In other words, suppose that

$$A \cdot \mathbf{z}_i = \mathbf{0}$$

for each  $i \in \{1, 2, ..., d\}$ . Then, any linear combination

 $\mathbf{z} = c_1 \mathbf{z}_1 + c_2 \mathbf{z}_2 + \dots + c_d \mathbf{z}_d$ 

is also a solution to the homogeneous linear-systems problem.

Theorem 35: Superposition for Solution to  $A \cdot \mathbf{x} = \mathbf{b}$ 

Suppose  $A \in \mathbb{R}^{m \times n}$  and  $\mathbf{b} \in \mathbb{R}^m$  are given, with  $\mathbf{b} \in \text{Span}\{A(:,k)\}_{k=1}^n$ . Suppose that  $\mathbf{x}^* \in \mathbb{R}^n$  is a particular solution to inhomogeneous linear systems

 $A\cdot \mathbf{x} = \mathbf{b}$ 

Then, any solution to our linear system problem can be written as

$$\mathbf{x} = \mathbf{x}^* + \mathbf{z}$$

where  $\mathbf{z}$  is any solution to the homogeneous linear system  $A \cdot \mathbf{x} = \mathbf{0}$ .

EXAMPLE 9.2.2

$$\mathbf{x}^* = \begin{bmatrix} 2.5\\15.0\\0.0 \end{bmatrix}$$

Further, if we want to find solutions to the homogeneous linear system  $A\mathbf{x} = 0$ , we can study the related problem

$$\begin{bmatrix} 1 & 0 & -1.25 \\ 0 & 1 & 3.00 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Notice that because of the special form of this matrix, we see immediately our solution

$$\mathbf{z}_1 = \begin{bmatrix} 1.25\\ -3.00\\ 1.00 \end{bmatrix}$$

We conclude that any solution to our original system should take the form

$$\mathbf{x} = \mathbf{x}^* + c_1 \mathbf{z}_1 = \begin{bmatrix} 2.5\\15.0\\0.0 \end{bmatrix} + c_1 \begin{bmatrix} 1.25\\-3.00\\1.00 \end{bmatrix}$$

In other words, there is no unique interpolating quadratic function.

**Remark:** Theorem 20 states that any linear combination of solutions to the homogeneous system  $A\mathbf{x} = \mathbf{0}$  also solves this problem. Thus, if we can find a maximal set of linearly independent vectors that solve the system  $A\mathbf{x} = \mathbf{0}$ , we can characterize ALL solutions to the homogeneous linear system  $A\mathbf{x} = \mathbf{0}$  as linear combination of these vectors. This is important in light of the next two theorems.

#### EXAMPLE 9.2.3

Suppose we are modeling the descent path for a Boeing 787 airplane landing in SFO. We can visualize this modeling problem as follows:



In this case, we choose to model the descent path using a cubic polynomial

$$a(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Here a(x) represents the altitude (in feet) of the airplane after it has travelled x miles in the horizontal direction. To determine the unknown coefficients, we impose the following conditions:

Condition	Verbal Description	Equation
i.	The landing point has an altitude of 0	a(0) = 0
ii.	The tangent line to the descent path is horizontal at landing	a'(0) = 0.
iii.	The cruising altitude is 40000 ft	a(400) = 40000

In other words, x = 0 miles at the landing point. We set x = 400 miles when the airplane begins its descent. Similarly,

We now work to create a system of 3 equations in 4 unknowns using the conditions above. Using our first equation a(0) = 0, we substitute x = 0 and a(0) = 0 to see

$$1 \cdot a_0 + 0 \cdot a_1 + 0 \cdot a_2 + 0 \cdot a_3 = 0$$

Next, let's move onto equation (ii.), which states a'(0) = 0. In this case, we have

$$0 \cdot a_0 + 1 \cdot a_1 + 2 \cdot 0 \cdot a_2 + 3 \cdot 0 \cdot a_3 = 0$$

Using the same reasoning, we translate condition (iii.) into the equation

$$1 \cdot a_0 + 400 \cdot a_1 + 400^2 \cdot a_2 + 400^3 \cdot a_3 = 40,000$$

vS20190403

We can now combine all three equations into the system

$1 \cdot a_0 + $	$0 \cdot a_1 + $	$0 \cdot a_2 + $	$0 \cdot a_3 =$	0
$0 \cdot a_0 + $	$1 \cdot a_1 + $	$0 \cdot a_2 + $	$0 \cdot a_3 =$	0
$1 \cdot a_0 + 4$	$400 \cdot a_1 + 16$	$0000 \cdot a_2 + 6400$	$00000 \cdot a_3 = 40$	000

Using our work in part a above, we can write this polynomial interpolation problem as follows:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 400 & 160000 & 64000000 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 40000 \end{bmatrix}}_{\mathbf{b}}$$

This is a general linear-systems problem  $A \mathbf{x} = \mathbf{b}$  with a  $3 \times 4$  coefficient matrix A. Just as before, our strategy to solve this problem is to transform the matrix A into RREF form using a sequence of elementary row transformations.



In other words, a solution to a given linear-systems problem  $A\mathbf{x} = \mathbf{b}$  if and only if the vector  $\mathbf{b}$  can be written as a linear combination of the columns of A.

The best way to find general solutions to linear-systems problems is to use row reduction into RREF and find solutions to our equivalent problem. In other words, to find the solution set for a linear system of equations, we:

1. Begin with our original equation

 $A \cdot \mathbf{x} = \mathbf{b}$ 

2. Reduce the coefficient matrix A into reduced row echelon form U = RREF(A)by multiplying A on the left by matrix E yielding

$$E \cdot A = U$$

where  $E = E_t \cdot E_{t-1} \cdots E_2 \cdot E_1$  is a product of elementary matrices  $E_1, E_2, ..., E_t$ and each elementary matrix  $E_j$  is either a shear matrix, a transposition matrix, or a dilation matrix for j = 1, 2, ..., t.

3. Simultaneously apply the same sequence of elementary matrices to the righthand side to produce new updated system

$$U \cdot \mathbf{x} = \mathbf{y}$$

where  $E\mathbf{b} = \mathbf{y}$ . As we will prove in the next section, the solution set to the equivalent system  $U \cdot \mathbf{x} = \mathbf{y}$  is identical to the solution set to the original system  $A \cdot \mathbf{x} = \mathbf{b}$ .

- 4. Decide if  $\mathbf{y} \in \text{Span}\{U(:,k)\}_{k=1}^n$ 
  - If  $\mathbf{y} \notin \text{Span}\{U(:,k)\}_{k=1}^n$ , then no exact solution exists to our original linear-system problem  $A \cdot \mathbf{x} = \mathbf{b}$ .
  - If  $\mathbf{y} \in \text{Span}\{U(:,k)\}_{k=1}^n$ , then find the general structure for any solution  $\mathbf{x}$  in the form

where  $\mathbf{x}^*$  is a particular solution to  $U \cdot \mathbf{x} = \mathbf{y}$  and  $\mathbf{z}$  is a general solution to the homogeneous system  $U \cdot \mathbf{x} = \mathbf{0}$ 

 $\mathbf{x} = \mathbf{x}^* + \mathbf{z}$ 

# Lesson 17: Solution sets to general linear-systems problem-Suggested Problems

1. Consider the following general linear-systems problem:



- **A.** Find all linearly independent solutions to this homogeneous equation  $A \cdot \mathbf{x} = \mathbf{0}$ . How many linearly independent solutions to  $A \cdot \mathbf{x} = \mathbf{0}$  are there? How does this relate to the number of nonpivot columns of U = RREF(A)
- **B.** Find a particular solution to GLSP.
- C. Find the entire solution set for this GLSP.
- 2. Consider the following general linear-systems problem



- **A.** Transform the general linear-systems problem into an equivalent system  $U \cdot \mathbf{x} = \mathbf{y}$  where U = RREF(A).
- **B.** What is our strategy to solve the general linear-systems problem?
- **C.** Find the solution set for this GLSP.
- **D.** Compare and contrast this strategy with the technique we used to solve the nonsingular linear-systems problem. How are these technique similar? How do these algorithms differ?