Answers **Lesson 16 Warm Up Quiz**

Math 2B: Applied Linear Algebra

True/False

For the problems below, circle T if the answer is true and circle F is the answer is false.

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1. **T** F If $A, B \in \mathbb{R}^{n\times n}$, $\det(AB) = \det(A)\det(B)$.

2. T **F** $\det[(A+B)(A-B)] = \det(A^2 - B^2)$.

3. **T** F If A is a square $n \times n$ matrix and $A^3 = 0$, then $\det(A) = 0$.

4. T **F** If A is invertible, then $\det\left(A^{-1}\right) = \det(\text{A})$.

5. **T** F If $\det(A) \neq 0$, then there will always be a solution to linear system $A\mathbf{x} = \mathbf{b}$.

6. T **F** If $A \in \mathbb{R}^{3\times 3}$, then $\det(5\,A) = 5\det(A)$

7. T **F** For square matrices $A, B \in \mathbb{R}^{n\times n}$, we have $\det(A+B) = \det(A) + \det(B)$.

8. **T** F If A is invertible, then $\det(A)\det\left(A^{-1}\right) = 1$.

9. **T** F If two rows of a 3×3 matrix A are the same, then $\det(A) = 0$.

10. T **F** For square $A \in \mathbb{R}^{n\times n}$, we have $\det(A^T) = -\det(A)$.

11. **T** F If $A \in \mathbb{R}^{n\times n}$ is nonsingular, then $\det(I) = \det(A)\det(A^{-1})$.

12. T **F** Any matrix $A^{m\times n}$ where $m > n$ with a zero row will have a zero determinant.

13. **T** F If $A = LU$ is the LU-Factorization of matrix $A \in \mathbb{R}^{n\times n}$, then $\det(A) = \det(U) = u_{11}u_{22}\cdots u_{nn}$.

14.	T	**F**	Suppose $A, B \in \mathbb{R}^{n \times n}$. If B is produced by interchanging two rows of A, then $\det(B) = \det(A)$.
15.	**T**	F	If A nonsingular, then $\det(A^{-1}) = \dfrac{1}{\det(A^T)}$.
16.	T	**F**	$\det(2A) = 2\ \det(\mathrm{A})$.
17.	**T**	F	If $B \in \mathbb{R}^{n \times n}$ is produced by multiplying row 3 of $A \in \mathbb{R}^{n \times n}$ by 5, then $\det(\mathrm{B}) = 5\ \det(\mathrm{A})$.
18.	**T**	F	$\det(A) = \det(A^T)$.
19.	T	**F**	If $\det(A) = 0$ for square matrix $A \in \mathbb{R}^{n \times n}$, then the corresponding system $A\mathbf{x} = \mathbf{b}$ will be inconsistent for all $\mathbf{b} \in \mathbb{R}^n$.
20.	**T**	F	If A nonsingular, then $\det(A^{-T}) = \dfrac{1}{\det(A)}$.
21.	**T**	F	If B is formed by adding to one row of A to a scalar multiple times another row of A, then $\det(B) = \det(A)$.
22.	T	**F**	For square $A \in \mathbb{R}^{n \times n}$, we have $\det(-A) = -\det(A)$.
23.	T	**F**	If A is an $n \times n$ matrix and $\det(A) = 2$, then $\det\left(A^3\right) = 6$.
24.	T	**F**	Any square matrix with all nonzero rows will have a nonzero determinant.
25.	T	**F**	Any system of n equations in n unknowns is consistent if and only if $\det(A) \neq 0$.
26.	T	**F**	If A and B are $n \times n$ matrices with $\det(A) = 2$ and $\det(B) = 3$, then $\det(A+B) = 5$.

27. **T** F For square $A \in \mathbb{R}^{n \times n}$, if $\det(A) \neq 0$, then A^{-1} exists.

28. **T** F If A is a 2×2 matrix with zero determinant, then one column of A is a scalar multiple of the other column.

29. T **F** $\det(I_n + A) = 1 + \det(A)$ for any $A \in \mathbb{R}^{n \times n}$.

30. **T** F Let $A \in \mathbb{R}^{4 \times 3}$ and $B \in \mathbb{R}^{3 \times 4}$. Then, the determinant of product AB must be zero.

31. T **F** For $A \in \mathbb{R}^{n \times n}$, if $\det(A) = 0$, then two rows or two columns of A are identical.

32. T **F** For $A \in \mathbb{R}^{n \times n}$, if $\det(A) = 0$, then a row or column of A is zero.

33. **T** F If A is a 2×2 matrix with zero determinant, then one column of A is a scalar multiple of the other column.

34. **T** F $\det(AB) = \det(B) \cdot \det(A)$.

35. **T** F If two rows of a 3×3 matrix A are the same, then $\det(A) = 0$.

36. **T** F If $\det(A) \neq 0$, then there will always be a solution to linear system $A\mathbf{x} = \mathbf{b}$.

37. T **F** If $A \in \mathbb{R}^{3 \times 3}$, then $\det(5\,A) = 15\det(A)$

38. T **F** Suppose $A, B \in \mathbb{R}^{2 \times 2}$. If $\det(A) = 2$ and $\det(B) = 3$, then $\det(A + B) = 5$.

Multiple Choice

For the problems below, circle the correct response for each question.

1. Let $A \in \mathbb{R}^{5\times 5}$ with $\det(A) = -12$. Suppose

$$B = S_{14}(5) \cdot P_{14} \cdot D_3(8) \cdot P_{23} \cdot D_3(1/4) \cdot A$$

where we use standard notation for elementary matrices as discussed in class. Then $\det(B)$ is

A. 0 B. 120 C. 24 **D. -24** E. -120

2. Let $A = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 0 & -2 \\ -3 & 1 & 0 \end{bmatrix}$. Then, using matrix-matrix multiplication, we see

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 3 & 2 \\ 2 & 0 & -2 \\ -3 & 1 & 0 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -13/3 \end{bmatrix}$$

Using the information above and your knowledge of the properties of determinants, which of the following gives $\det(A)$:

A. $\det(A) = -39$
B. $\det(A) = -13$
C. $\det(A) = 13$
D. $\det(A) = 26$
E. $\det(A) = -26$

3. Suppose that $A \in \mathbb{R}^{n\times n}$ has nonzero determinant. Which of the following is NOT true:

A. $\dim(\mathrm{Nul}(A)) > 0$
B. $\mathrm{Nul}(A) = \mathrm{Nul}(A^T)$
C. $\mathrm{Col}(A) = \mathrm{Col}(A^T)$
D. $\mathrm{rank}(A) \leq n$
E. None of these

4. Recall that any determinant function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a map from the set of $n \times n$ matrices to the real numbers such that

$$\det(A) = 0 \text{ if } A \text{ singular,}$$
$$\det(A) \neq 0 \text{ if } A \text{ nonsingular.}$$

Which of the following is not one of the five properties that such a determinant function must satisfy

A. $\det(I_n) = 1$.
B. $\det(A) = 0$ if $A \in \mathbb{R}^{n\times n}$ has a row of all zero entries.
C. $\det(P_{ik} \cdot A) = -\det(A)$ for any transposition matrix $P_{ik} \in \mathbb{R}^{n\times n}$.
D. $\det(S_{ik}(c) \cdot A) = c \cdot \det(A)$ for all $1 \leq i \leq n, 1 \leq k \leq n, i \neq k$ and $c \in \mathbb{R}$.
E. $\det(D_j(c) \cdot A) = c \cdot \det(A)$ for each $1 \leq j \leq n$ and all $c \in \mathbb{R}$.

5. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & -2 & 0 \end{bmatrix}$. Then, which of the following gives $\det(A)$:

A. 0 B. 10 C. 6 **D. -6** E. None of these.

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6. Suppose that $B \in \mathbb{R}^{3\times 3}$ with the property that $\det(B^2) = \det(B)$. Which of the following statements about B must be true:

A. B is invertible

B. $\det(B) = 0$

C. $B = B^2$

D. $\det\left(B^5\right) = \det\left(B^3\right)$

E. None of these

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7. Suppose that $U \in \mathbb{R}^{3\times 3}$ is the upper triangular matrix from the LU factorization of matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 5 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

in problem 6 above. What do you know about the product of the diagonal elements of U given by $u_{11}u_{22}u_{33}$?

A. $\det(A) = a_{11}a_{22}a_{33}$ B. $u_{11}u_{22}u_{33} = 0$ C. $u_{11}u_{22}u_{33} = 1$

D. $u_{11}u_{22}u_{33} = -30$ E. $u_{11}u_{22}u_{33} = 30$

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8. Let $A \in \mathbb{R}^{5\times 5}$ with $\det(A) = -4$. Suppose

$$S_{14}(4) \cdot P_{14} \cdot D_3(1/8) \cdot P_{23} \cdot D_3(4) \cdot P_{12} \cdot B = A$$

where we use standard notation for elementary matrices as discussed in class. Then $\det(B)$ is

A. 0 B. 2 C. -2 **D. 8** E. -8

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9. Suppose that $A \in \mathbb{R}^{3\times 3}$ with inverse given by

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, find $\det(A)$:

A. $\frac{1}{6}$ B. $-\frac{1}{6}$ **C. -6** D. 6 E. $\frac{2}{3}$

Free Response

1. For what values of a, b, c is the matrix $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ invertible?

2. Use the formula for the determinant of an 3×3 to prove that the determinant of a lower triangular matrix is the product of the diagonal elements.

3. Prove the 2×2 determinant function that satisfies the five properties we discussed in class must be unique?

   A. $\det(A) = \det\left(A^T\right)$
   B. If $i \neq k$, then $\det(P_{ik} \cdot A) = -\det(A)$
   C. If $i \neq k$, then $\det(S_{ik}(c) \cdot A) = \det(A)$
   D. For $1 \leq i \leq n$, $\det(D_i(c) \cdot A) = c \cdot \det(A)$
   E. . $\det(A \cdot B) = \det(A) \cdot \det(B)$

4. Prove the 3×3 determinant function that satisfies the five properties we discussed in class must be unique?

   A. $\det(A) = \det\left(A^T\right)$
   B. If $i \neq k$, then $\det(P_{ik} \cdot A) = -\det(A)$
   C. If $i \neq k$, then $\det(S_{ik}(c) \cdot A) = \det(A)$
   D. For $1 \leq i \leq n$, $\det(D_i(c) \cdot A) = c \cdot \det(A)$

5. Use the general formula for determinants to prove $\det(A) = \det\left(A^T\right)$

6. Recall that for $A \in \mathbb{R}^{3\times 3}$, the determinant of A was given by

$$\det(A) = \sum_{\pi \in S_3} \operatorname{sgn}(\pi)\, a_{\pi(1),1}\, a_{\pi(2),2}\, a_{\pi(3),3}$$

(a) List all permutations $\pi \in S_3$. In other words, list all maps $\pi : \{1,2,3\} \to \{1,2,3\}$ that are one-to-one and onto.

Solution: Consider S_3, the permutation group on a set with three elements. We know that for each $i \in \{1, 2, ..., 6\}$, we have permutations

$$\pi_i : \{1,2,3\} \longrightarrow \{1,2,3\}$$

that are both one-to-one and onto. From our theorem above, we know that S_3 contains exactly $3! = 6$ different permutations. We will label these permutations here. Consider:

$$\pi_1 := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \qquad \pi_2 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad \pi_3 := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\pi_4 := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad \pi_5 := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad \pi_6 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

To investigate inversions with respect to each π_i, we need to consider three different pairs:

$$(1,2) \qquad\qquad (1,3) \qquad\qquad (2,3)$$

We see that $\operatorname{Inv}(\pi_i) = \{(1,3),(2,3)\}$ are given by

$$\begin{array}{lcl}
\operatorname{Inv}(\pi_1) = \emptyset & \Longrightarrow & n(\pi_1) = 0 \\
\operatorname{Inv}(\pi_2) = \{(1,3),(2,3)\} & \Longrightarrow & n(\pi_2) = 2 \\
\operatorname{Inv}(\pi_3) = \{(1,2),(1,3)\} & \Longrightarrow & n(\pi_3) = 2 \\
\operatorname{Inv}(\pi_4) = \{(1,2),(1,3),(2,3)\} & \Longrightarrow & n(\pi_4) = 3 \\
\operatorname{Inv}(\pi_5) = \{(2,3)\} & \Longrightarrow & n(\pi_5) = 1 \\
\operatorname{Inv}(\pi_6) = \{(1,2)\} & \Longrightarrow & n(\pi_6) = 1
\end{array}$$

We can use this data to confirm that

$$\begin{aligned}
\operatorname{sgn}(\pi_1) &= (-1)^{n(\pi_1)} = (-1)^0 = +1 \\
\operatorname{sgn}(\pi_2) &= (-1)^{n(\pi_2)} = (-1)^2 = +1 \\
\operatorname{sgn}(\pi_3) &= (-1)^{n(\pi_3)} = (-1)^2 = +1 \\
\operatorname{sgn}(\pi_4) &= (-1)^{n(\pi_4)} = (-1)^3 = -1 \\
\operatorname{sgn}(\pi_5) &= (-1)^{n(\pi_5)} = (-1)^1 = -1 \\
\operatorname{sgn}(\pi_6) &= (-1)^{n(\pi_6)} = (-1)^1 = -1
\end{aligned}$$

With this, we are ready to build the determinant function for 3×3 matrices.

(b) Use your work in part (a) and the determinant formula given above to prove that that the determinant of an upper triangular matrix $U \in \mathbb{R}^{3\times 3}$ is the product of the main diagonal elements.

Solution: Using the information above, for matrix $A \in \mathbb{R}^{3\times 3}$ we have

$$\det(A) = \sum_{\pi \in S_n} \text{sgn}(\pi) \cdot a_{1\pi(1)} \cdot a_{2\pi(2)} \cdot a_{3\pi(3)}$$

$$= \sum_{i=1}^{6} \text{sgn}(\pi_i) \cdot a_{1\pi_i(1)} \cdot a_{2\pi_i(2)} \cdot a_{3\pi(3)}$$

$$= a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32}$$
$$- a_{13} \cdot a_{22} \cdot a_{31} - a_{11} \cdot a_{23} \cdot a_{32} - a_{12} \cdot a_{21} \cdot a_{33}$$

If we assume that A is upper-triangular, we know

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \Longleftrightarrow \qquad a_{ik} \begin{cases} \in \mathbb{R} & \text{if } i \leq k \\ = 0 & \text{if } i > k \end{cases}$$

Thus, since all permutations $\pi_2, \pi_3, ..., \pi_6$ contain at least one inversion, we see that the determinant function greatly simplifies to

$$\det(A) = a_{11} \cdot a_{22} \cdot a_{33}$$

which is just the product of the diagonal elements. This is exactly what we wanted to show.