### 1.7 Permutations

## Definition 1.16: Permutation of a Set $A$

Let $A$ be a nonempty set. A permutation of $\mathbf{A}$ is a bijective function from $A$ to itself. In other words, a permutation is a map

$$
\pi: A \longrightarrow A
$$

such that
i. $\pi$ is one-to-one: if $\pi(a)=\pi(b)$, then $a=b$
ii. $\pi$ is onto: Codomain $(\pi)=\operatorname{Rng}(\pi)$

Permutations are the mathematical technology used to discuss rearrangements of the ordering of a specific set.

## EXAMPLE 1.7.1

Imagine we want to rotate tires on a 2006 Honda Civic during a routine tire change at America's tires. At the beginning of our job, each of the four tires is already on the car in one of the four positions. We define the set $A=\{1,2,3,4\}$ used to label each tire location and assign each tire in that location a number in $A$. We do this by sitting in the drivers seat, and labeling the front left tire number one, the front right number 2 , the back left tire number three and the back right tire number four.

To rearrange the tires, we want to place each tire in one of the four positions on the car and no two tires can be place in the same location. One possible rearrangement is to define permutation $f: A \rightarrow A$ on the set $A=\{1,2,3,4$.$\} We can use a$ mathematic description net of $A$ under our map $f$ :

$$
f(1)=4, \quad f(2)=3, \quad f(3)=2, \quad f(4)=1
$$

Here, we've assigned switched the tires in front left and back right positions and we've switched the tires in the front right and back right.

Our mapping $f$ is injective since each $i \in A$ has a unique image under $f$. We see that $f$ is surjective since $\operatorname{Rng}(f)=A$.

## Definition 1.17: The set of all permutations of $n$ elements $\left(S_{n}\right)$

Let $[n]=\{j \in \mathbb{N}: 1 \leq j \leq n\}$. The permutation group on [ $n$ ], denoted as $S_{n}$, is the set of all permutations of the set $[n]=\{1,2, \ldots, n\}$. These are exactly the set of one-to-one maps from the set of the first $n$ integers to itself. For any element $\pi \in S_{n}, \pi$ maps $\{1,2, \ldots, n\}$ onto itself.

Cauchy's two-line notation for a permutation of the integers

$$
\pi:\{1,2, \ldots, n\} \longrightarrow\{1,2, \ldots, n\}
$$

is given as:

$$
\left(\begin{array}{ccccc}
1 & 2 & 3 & \cdots & n \\
\pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n)
\end{array}\right)
$$

In this notation, we can easily see that the image of $i$ is $\pi(i)$ under the map $\pi$.

## EXAMPLE 1.7.2

Suppose you own a limousine company in San Francisco. You have six employees that work the night shift from $4 \mathrm{pm}-2 \mathrm{am}$. You also have six different limousines, each with a different make and model. Because your employees tend to bicker about driving assignments, you decide to randomize the picking process using permutations. You give each of your employees a number from in the set $\{1,2,3,4,5,6\}$ and you do the same for your fleet of cars. Each afternoon before work, you assign each driver to a car by determining a map between the driver number and the car number. For example, you determine the following map:

$$
\pi(1)=5, \quad \pi(2)=2, \quad \pi(3)=1, \quad f(4)=3 . \quad \pi(5)=6, \quad f(6)=4
$$

While the function notation provides a clear mapping, it is much more efficient to track the entire mapping all at once. We can do this via a two row, six column array:

$$
\left(\begin{array}{llllll}
1 & 2 & 3 & 4 & 5 & 6 \\
5 & 2 & 1 & 3 & 6 & 4
\end{array}\right)
$$

The first row of our array above indicates the driver number and the second row assigns each driver to the car they will drive that night.

Under this permutation, we see that driver 2 is assigned car number 2 while driver 6 is assigned car 4. You post the day's permutation on the board before work so that all employees can quickly and easily determine which car they are assigned.

## Theorem 6: The Number of Elements of $S_{n}$

Let $[n]=\{1,2, \ldots, n\}$. If $S_{n}$ is the set of all bijections from $[n]$ to $[n]$, then there are a total of $n!=n \cdot(n-1) \cdots 2 \cdot 1$ elements of $S_{n}$.

Proof. Let $\pi:[n] \rightarrow[n]$ be a permutation.

## EXAMPLE 1.7.3

Let's list all the permutations of $S_{2}$, the permutation group on a set with two elements. By our definition of $S_{2}$, we want to find all bijective maps

$$
\pi:\{1,2\} \longrightarrow\{1,2\}
$$

We know by our theorem above that there are precisely $2!=2$ such permutations. Let's list these:

$$
\pi_{1}:=\left(\begin{array}{ll}
1 & 2 \\
1 & 2
\end{array}\right), \quad \pi_{2}:=\left(\begin{array}{ll}
1 & 2 \\
2 & 1
\end{array}\right)
$$

## EXAMPLE 1.7.4

Consider $S_{3}$, the permutation group on a set with three elements. From our theorem above, we know that $S_{3}$ contains exactly $3!=6$ different permutations. We will label these permutations here. Consider:

$$
\begin{array}{ll}
\pi_{1}:=\left(\begin{array}{lll}
1 & 2 & 3 \\
1 & 2 & 2
\end{array}\right), & \pi_{2}:=\left(\begin{array}{lll}
1 & 2 & 3 \\
2 & 3 & 1
\end{array}\right),
\end{array}
$$

We will come back to this permutation often in our discussion of the signs of any permutation as well in our development of the determinant in Chapter 7.

## Definition 1.18: Inversion of a pair $(i, j)$ with respect to $\pi$

Let $\pi \in S_{n}$. Suppose that $i, j \in[n]$ are chosen such that

$$
1 \leq i<j \leq n
$$

We call the pair $(i, j)$ an inversion with respect to $\pi$ if and only if

$$
\pi(i)>\pi(j)
$$

In other words, we call the pair $(i, j)$ and inversion with respect to a given permutation $\pi$ if the images $\pi(i)$ and $\pi(j)$ are in have opposite orders from the pre-images $i$ and $j$. One quick way to checki if $(i, j)$ is an inversion with respect to $\pi$ is to calculate the ratio

$$
\frac{\pi(i)-\pi(j)}{i-j}
$$

If this ratio is less than 0 , it means that $(i, j)$ is an inversion with respect to $\pi$. On the other hand, if this ratio is greater than 0 , then the pair $(i, j)$ is not an inversion with respect to $\pi$.

## EXAMPLE 1.7 .5

Consider $S_{2}=\left\{\pi_{1}, \pi_{2}\right\}$. In this case, because $\pi_{i}:\{1,2\} \longrightarrow\{1,2\}$, we only have to check the pair $(1,2)$. We begin with $\pi_{1}$. Consider

$$
\frac{\pi_{1}(1)-\pi_{1}(2)}{1-2}=\frac{1-2}{1-2}=\frac{-1}{-1}=1>0
$$

Since this ratio is positive, we know $(1,2)$ is NOT an inversion with respect to $\pi_{1}$. On the other hand, let's consider the pair $(1,2)$ with respect to the permutation $\pi_{2}$. We notice that

$$
\frac{\pi_{2}(1)-\pi_{2}(2)}{1-2}=\frac{2-1}{1-2}=\frac{1}{-1}=-1<0
$$

Since this ratio is negative, we know that $(1,2)$ is an inversion with respect to $\pi_{2}$.

## Definition 1.19: Set of all inversions for a given $\pi \in S_{n}$

Let $\pi \in S_{n}$ be given. Then we define the set of all inversions with respect to $\pi$ as

$$
\operatorname{Inv}(\pi)=\{(i, j): i, j \in[n], i<j, \text { and } \pi(i)>\pi(j)\}
$$

## EXAMPLE 1.7.6

Let's consider the set $S_{3}$, which has a total of $3!=6$ permutations. We know that for each $i \in\{1,2, \ldots, 6\}$, we have

$$
\pi_{i}:\{1,2,3\} \longrightarrow\{1,2,3\}
$$

Thus, if we are going to investigate inversions with respect to each $\pi_{i}$, we need to consider three different pairs:

$$
(1,2)
$$

$$
\begin{equation*}
(1,3) \tag{1,2}
\end{equation*}
$$

For each of these permutations, we can find $\operatorname{Inv}\left(\pi_{i}\right)$. We begin with the identity permutation $\pi_{1}$, and we note that $\operatorname{Inv}\left(\pi_{1}\right)=\emptyset$ since

$$
\begin{aligned}
& \frac{\pi_{1}(1)-\pi_{1}(2)}{1-2}=\frac{1-2}{1-2}>0 \\
& \frac{\pi_{1}(1)-\pi_{1}(3)}{1-3}=\frac{1-3}{1-2}>0 \\
& \frac{\pi_{1}(2)-\pi_{1}(3)}{2-3}=\frac{2-3}{1-2}>0
\end{aligned}
$$

Because there are no inversions with respect to $\pi_{1}$, we confirm that $\operatorname{Inv}\left(\pi_{i}\right)=\emptyset$. Next, let's move onto $\pi(2)$. We can analyze each pair $(i, j)$ with respect to $\pi_{2}$ to find:

$$
\begin{aligned}
& \frac{\pi_{2}(1)-\pi_{2}(2)}{1-2}=\frac{2-3}{1-2}>0 \\
& \frac{\pi_{2}(1)-\pi_{2}(3)}{1-3}=\frac{2-1}{1-2}<0 \\
& \frac{\pi_{2}(2)-\pi_{2}(3)}{2-3}=\frac{3-1}{1-2}<0
\end{aligned}
$$

Thus, we see that $\operatorname{Inv}\left(\pi_{2}\right)=\{(1,3),(2,3)\}$ has two elements. We can continue in this manner to confirm

$$
\begin{aligned}
& \operatorname{Inv}\left(\pi_{1}\right)=\emptyset \\
& \operatorname{Inv}\left(\pi_{2}\right)=\{(1,3),(2,3)\} \\
& \operatorname{Inv}\left(\pi_{3}\right)=\{(1,2),(1,3)\} \\
& \operatorname{Inv}\left(\pi_{4}\right)=\{(1,2),(1,3),(2,3)\} \\
& \operatorname{Inv}\left(\pi_{5}\right)=\{(2,3)\} \\
& \operatorname{Inv}\left(\pi_{6}\right)=\{(1,2)\}
\end{aligned}
$$

## Definition 1.20: Sign of a Permutation

Using this definition, we can create a function $n: S_{n} \rightarrow[n]$ such that $n(\pi)$ gives the number of distinct inversions with respect to $\pi$. Using this output, we can define the sign of a permutation $\pi$ to be $\operatorname{sgn}(\pi)=$ $(-1)^{n(\pi)}$. We can write this another way:
$\operatorname{sgn}(\pi)= \begin{cases}+1 & \text { if there are an even number of inversions with respect to } \pi, \\ -1 & \text { if there are an odd number of inversions with respect to } \pi .\end{cases}$

## EXAMPLE 1.7.7

Let's find the sign of each permutation in $S_{3}$. We do so by recalling that

$$
\begin{array}{lll}
\operatorname{Inv}\left(\pi_{1}\right)=\emptyset & \Longrightarrow & n\left(\pi_{1}\right)=0 \\
\operatorname{Inv}\left(\pi_{2}\right)=\{(1,3),(2,3)\} & \Longrightarrow & n\left(\pi_{2}\right)=2 \\
\operatorname{Inv}\left(\pi_{3}\right)=\{(1,2),(1,3)\} & \Longrightarrow & n\left(\pi_{3}\right)=2 \\
\operatorname{Inv}\left(\pi_{4}\right)=\{(1,2),(1,3),(2,3)\} & \Longrightarrow & n\left(\pi_{4}\right)=3 \\
\operatorname{Inv}\left(\pi_{5}\right)=\{(2,3)\} & \Longrightarrow & n\left(\pi_{5}\right)=1 \\
\operatorname{Inv}\left(\pi_{6}\right)=\{(1,2)\} & \Longrightarrow & n\left(\pi_{6}\right)=1
\end{array}
$$

We can use this data to confirm that

$$
\begin{aligned}
& \operatorname{sgn}\left(\pi_{1}\right)=(-1)^{n\left(\pi_{1}\right)}=(-1)^{0}=+1 \\
& \operatorname{sgn}\left(\pi_{2}\right)=(-1)^{n\left(\pi_{2}\right)}=(-1)^{2}=+1 \\
& \operatorname{sgn}\left(\pi_{3}\right)=(-1)^{n\left(\pi_{3}\right)}=(-1)^{2}=+1 \\
& \operatorname{sgn}\left(\pi_{4}\right)=(-1)^{n\left(\pi_{4}\right)}=(-1)^{3}=-1 \\
& \operatorname{sgn}\left(\pi_{5}\right)=(-1)^{n\left(\pi_{5}\right)}=(-1)^{1}=-1 \\
& \operatorname{sgn}\left(\pi_{6}\right)=(-1)^{n\left(\pi_{6}\right)}=(-1)^{1}=-1
\end{aligned}
$$

This will come in very helpful in our development of the determinant function in Chapter 7.

## Definition 1.21: Cycles

Let $x_{1}, x_{2}, \ldots, x_{k} \in[n]$ such that $x_{i} \neq x_{j}$ for $i, j \in[k]$ with $k \leq n$ and $i \neq j$. A $k$-cycle is a permutation $f$ of the form

$$
\pi\left(x_{1}\right)=x_{2}, \quad \pi\left(x_{2}\right)=x_{3}, \quad \ldots, \quad \pi\left(x_{k-1}\right)=x_{k}, \quad \pi\left(x_{k}\right)=x_{1}
$$

where $\pi(i)=i$ for any $i \in S_{n}-\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$.

One of the really easy ways to represent cycles is via permutation diagram that illustrates the relationships between permutation elements using vertices and nodes.

## EXAMPLE 1.7 .8

Lets consider the permutation $\pi \in S_{8}$ defined as follows:

$$
\pi:=\left(\begin{array}{llllllll}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
5 & 2 & 8 & 1 & 3 & 6 & 7 & 4
\end{array}\right) .
$$

This permutation is a 5 cycle as we see here ( $\left.1 \begin{array}{llll}5 & 3 & 8 & 4\end{array}\right)$. We can also visualize the permutation using a permutation diagram.


This diagram will come in very useful if we'd like to learn more about the composition of two permutations $\sigma, \pi \in S_{n}$.

## EXAMPLE 1.7 .9

The set of permutations of $A=\{1,2,3\}$ can be written out in a multitude of ways including an explicit description of the individual function values, the matrix form or the cycle decomposition form.


This diagram will come in very useful if we'd like to learn more about the composition of two permutations $\sigma, \pi \in S_{n}$.

Theorem 7: Existence of a Cycle Decomposition of Permutations

Let $\pi:[n] \rightarrow[n]$ be a permutation. Then, $\pi$ can be written using a unique cycle decomposition.

## Theorem 8: Transpositions Generate All Permutations

Every element of $S_{n}$ may be written as a composition of transpositions.

Proof. Use logic set up in Dummit and Foote.


