Name : ____________________ **Class Number**: _____

Permutation Definition of Determinant

Studying permutation group S_2

Let's list all the permutations of S_2, the permutation group on a set with two elements. By our definition of S_2, we want to find all bijective maps

$$\pi : \{1,2\} \longrightarrow \{1,2\}$$

We know by our theorem, there are precisely $2! = 2$ such permutations. Let's list these:

$$\pi_1 := \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \qquad \pi_2 := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Finding inversion pairs for S_2

Let's now look at the only pair of two elements $(1, 2)$ and study the inversions of this pair with respect to each π_i. To this end, consider:

$$\frac{\pi_1(1) - \pi_1(2)}{1 - 2} =$$

$$\frac{\pi_2(1) - \pi_2(2)}{1 - 2} =$$

We can continue in this manner to confirm

$$\operatorname{Inv}(\pi_1) =$$

$$\operatorname{Inv}(\pi_2) =$$

Studying permutation group S_3

Consider S_3, the permutation group on a set with three elements. From our theorem, we know S_3 contains exactly $3! = 6$ different permutations. We will label these permutations below:

$$\pi_1 := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \qquad \pi_2 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad \pi_3 := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\pi_4 := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad \pi_5 := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad \pi_6 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Finding inversion pairs for S_3

If we are going to investigate inversions with respect to each π_i, we need to consider three different pairs:

$$(1, 2) \qquad (1, 3) \qquad (2, 3)$$

For each of these permutations, we can find $\operatorname{Inv}(\pi_i)$.

We begin with the identity permutation π_1 and analyze each pair (i, j) with respect to π_1:

$$\frac{\pi_1(1) - \pi_1(2)}{1 - 2} =$$

$$\frac{\pi_1(1) - \pi_1(3)}{1 - 3} =$$

$$\frac{\pi_1(2) - \pi_1(3)}{2 - 3} =$$

We can analyze each pair (i, j) with respect to π_2 to find:

$$\frac{\pi_2(1) - \pi_2(2)}{1 - 2} =$$

$$\frac{\pi_2(1) - \pi_2(3)}{1 - 3} =$$

$$\frac{\pi_2(2) - \pi_2(3)}{2 - 3} =$$

Let's analyze each pair (i, j) with respect to π_3:

$$\frac{\pi_3(1) - \pi_3(2)}{1 - 2} =$$

$$\frac{\pi_3(1) - \pi_3(3)}{1 - 3} =$$

$$\frac{\pi_3(2) - \pi_3(3)}{2 - 3} =$$

$$\frac{\pi_4(1) - \pi_4(2)}{1 - 2} =$$

$$\frac{\pi_4(1) - \pi_4(3)}{1 - 3} =$$

$$\frac{\pi_4(2) - \pi_4(3)}{2 - 3} =$$

$$\frac{\pi_5(1) - \pi_5(2)}{1 - 2} =$$

$$\frac{\pi_5(1) - \pi_5(3)}{1 - 3} =$$

$$\frac{\pi_5(2) - \pi_5(3)}{2 - 3} =$$

$$\frac{\pi_6(1) - \pi_6(2)}{1 - 2} =$$

$$\frac{\pi_6(1) - \pi_6(3)}{1 - 3} =$$

$$\frac{\pi_6(2) - \pi_6(3)}{2 - 3} =$$

Thus, let's find the set of inversion pairs and the number of inversions with respect to each permutation and write these below:

$\text{Inv}\,(\pi_1) = \qquad \Rightarrow \qquad n(\pi_1) =$

$\text{Inv}\,(\pi_2) = \qquad \Rightarrow \qquad n(\pi_2) =$

$\text{Inv}\,(\pi_3) = \qquad \Rightarrow \qquad n(\pi_3) =$

$\text{Inv}\,(\pi_4) = \qquad \Rightarrow \qquad n(\pi_4) =$

$\text{Inv}\,(\pi_5) = \qquad \Rightarrow \qquad n(\pi_5) =$

$\text{Inv}\,(\pi_6) = \qquad \Rightarrow \qquad n(\pi_6) =$

Finally, we can use our work above to find the sign of each permutation in S_3:

$\text{Sgn}\,(\pi_1) =$

$\text{Sgn}\,(\pi_2) =$

$\text{Sgn}\,(\pi_3) =$

$\text{Sgn}\,(\pi_4) =$

$\text{Sgn}\,(\pi_5) =$

$\text{Sgn}\,(\pi_6) =$