Chapter 7

Determinants

7.1 The Determinant Function

Let's consider a special case of the linear-systems problem. Given a square matrix $A \in \mathbb{R}^{n \times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$, find a vector $\mathbf{x}$ such that

$$A \cdot \mathbf{x} = \mathbf{b}.$$

Recall by our discussion of solutions sets for linear-systems problems, we can classify the solutions to this system into three situations

i. No exact solution: $\mathbf{b} \notin \text{Span}\{A(:,k)\}_{k=1}^{n}$

ii. One, unique solution: $\mathbf{b} \notin \text{Span}\{A(:,k)\}_{k=1}^{n}$ and the columns of A are linearly independent

iii. Non-unique solutions: $\mathbf{b} \notin \text{Span}\{A(:,k)\}_{k=1}^{n}$ and the columns of A are linearly dependent

For square matrices, we can very quickly determine which situations may arise by studying the matrix A. By the Invertible Matrix Theorem, we know that if A^{-1} exists, then the solution to

$$A \cdot \mathbf{x} = \mathbf{b}.$$

will always exist and it will be unique. Thus, if we can find a way to quickly identify whether a given square matrix A is invertible, we can immediately ascertain a lot of information about the associated linear system.

One possible mechanism we can use to determine if matrix A is invertible is to transform A into reduced-row echelon form. However, because this requires lots of arithmetic and proves to be very expensive for large matrices, we might want to develop some other techniques to determine if a square matrix is invertible.

The determinant function is one such tool. We use the determinant function as an oracle that translates the problem of identifying whether a given $n \times n$ matrix is invertible into a problem of checking to see if a number is equal to zero (or not). For any $n \in \mathbb{N}$ and $A \in \mathbb{R}^{n \times n}$ we want to create a function

$$\det\colon \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}, \qquad \begin{cases} \det(A) & \neq 0 \text{ if } A \text{ is invertible,} \\ \det(A) & = 0 \text{ if } A \text{ is singular.} \end{cases}$$

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Let $n \in \mathbb{N}$. A function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a determinant function if and only if it satisfies the following conditions

i. $\det(I_n) = 1$ where I_n is the $n \times n$ identify matrix

ii. If $A \in \mathbb{R}^{n\times n}$ has an all zero row, then $\det(A) = 0$

iii. $\det(S_{ik}(c) \cdot A) = \det(A)$

iv. $\det(P_{ik} \cdot A) = -\det(A)$

v. $\det(D_i(c) \cdot A) = c \cdot \det(A)$

As we will see, there is a unique function that satisfies these conditions.

Theorem 30: Permutation Definition of Determinant

Let $A \in \mathbb{R}^{n\times n}$. Then there is a unique determinant function

$$\det : \mathbb{R}^{n\times n} \to \mathbb{R}$$

given by

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \cdot a_{1\pi(1)} \cdot a_{2\pi(2)} \cdots a_{n\pi(n)}$$

EXAMPLE 7.1.1

Let's consider the $n = 2$ case for finding the determinant of a 2×2 matrix. We recall that S_2, the set of all permutations of $\pi : \{1,2\} \to \{1,2\}$ had only two elements, given in Cauchy's two-line notation as

$$\pi_1 := \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \qquad \pi_2 := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then, for any 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

we see

$$\begin{aligned}
\det(A) &= \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \cdot a_{1\pi(1)} \cdot a_{2\pi(2)} \\
&= \sum_{i=1}^{2} \operatorname{sgn}(\pi_i) \cdot a_{1\pi_i(1)} \cdot a_{2\pi_i(2)} \\
&= \operatorname{sgn}(\pi_1) \cdot a_{1\pi_1(1)} \cdot a_{2\pi_1(2)} + \operatorname{sgn}(\pi_2) \cdot a_{1\pi_2(1)} \cdot a_{2\pi_2(2)} \\
&= a_{11} \cdot a_{22} - a_{12} \cdot a_{21}
\end{aligned}$$

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EXAMPLE 7.1.2

Let's consider the $n = 3$ case for finding the determinant of a 3×3 matrix. We recall that S_3, the set of all permutations of $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$ had six elements, given in Cauchy's two-line notation as

$$\pi_1 := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \qquad \pi_2 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad \pi_3 := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\pi_4 := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad \pi_5 := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad \pi_6 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Recall also from our discussion of the signs of a permutation, we know that

$$\operatorname{sgn}(\pi_1) = 1, \qquad \operatorname{sgn}(\pi_2) = 1, \qquad \operatorname{sgn}(\pi_3) = 1,$$

$$\operatorname{sgn}(\pi_4) = -1, \qquad \operatorname{sgn}(\pi_5) = -1, \qquad \operatorname{sgn}(\pi_6) = -1.$$

Then, using this information, we know that for any 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

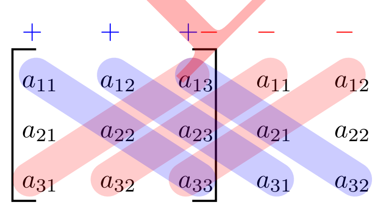
we can find the determinant as

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \cdot a_{1\pi(1)} \cdot a_{2\pi(2)} \cdot a_{3\pi(3)}$$

$$= \sum_{i=1}^{6} \operatorname{sgn}(\pi_i) \cdot a_{1\pi_i(1)} \cdot a_{2\pi_i(2)} \cdot a_{3\pi(3)}$$

$$\begin{aligned} &= a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32} \\ &- a_{13} \cdot a_{22} \cdot a_{31} - a_{11} \cdot a_{23} \cdot a_{32} - a_{12} \cdot a_{21} \cdot a_{33} \end{aligned}$$

The permutation definition of the determinant is the unique description of the determinant function. However, this definition is also extremely difficult to work with in practice. To reconcile this, mathematicians have developed a number of practical tricks to compute determinants of small matrices. For 3×3 matrices, we can use the following diagram to quickly calculate the determinant.



We multiply the three digits along the diagonals together. We add the product of each blue diagonal and subtract the product of each red diagonal to find the determinant of the 3×3 matrix.

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Theorem 31: Properties of Determinants

Let $A, B \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}$. Then the determinant function $\det(A)$ given by

$$\det(A) = \sum_{\pi\in S_n} \operatorname{sgn}(\pi)\cdot a_{1\pi(1)}\cdot a_{2\pi(2)}\cdots a_{n\pi(n)}$$

satisfies all of the following:

1. If A is upper- or lower-triangular, then $\det(A) = a_{11}\cdot a_{22}\cdots a_{nn}$
2. $\det(A) = \det\left(A^T\right)$
3. $\det(A\cdot B) = \det(A)\cdot\det(B)$
4. If S invertible, then $\det\left(S\cdot A\cdot S^{-1}\right) = \det(A)$
5. If $i \neq k$, then $\det(P_{ik}\cdot A) = -\det(A)$
6. If $i \neq k$, then $\det(S_{ik}(c)\cdot A) = \det(A)$
7. For $1 \leq i \leq n$, $\det(D_i(c)\cdot A) = c\cdot\det(A)$
8. $\det(cA) = c^n\det(A)$
9. A is invertible if and only if $\det(A) \neq 0$

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Lesson 18: Determinants- Suggested Problems

1. Use the formula for the determinant of a 3×3 matrix to find the determinant of the matrix
$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

2. Let $c \in \mathbb{R}$ be nonzero. Use the formula for the determinant of a 2×2 matrix to find the determinants of each of the following matrices
$$S_{21}(c) = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}, \qquad S_{12}(c) = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.$$

3. Let $c \in \mathbb{R}$ be nonzero. Use the formula for the determinant of a 2×2 matrix to find the determinants of each of the following matrices
$$D_1(c) = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \qquad D_2(c) = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}.$$

4. Use the formula for the determinant of a 2×2 matrix to find the determinants of the following matrix
$$P_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

5. Let $c \in \mathbb{R}$ be nonzero. Use the formula for the determinant of a 3×3 matrix to find the determinant of the following matrices
$$S_{21}(c) = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_{31}(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix}, \quad S_{32}(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix},$$
$$S_{12}(c) = \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_{13}(c) = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_{23} = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Let $c \in \mathbb{R}$ be nonzero. Use the formula for the determinant of a 3×3 matrix to find the determinant of the following matrices
$$D_1(c) = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_2(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_3(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}.$$

7. Use the formula for the determinant of a 3×3 matrix to find the determinant of the following matrices
$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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8. Use the formula for the determinant of a 2×2 matrix to to prove that the determinant of an upper-triangular matrix $U \in \mathbb{R}^{2\times 2}$ is the product of the diagonal elements. In other words, prove that

$$\det(U) = \det\left(\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}\right) = u_{11}\, u_{22}.$$

9. Use the formula for the determinant of a 3×3 matrix to to prove that the determinant of an upper-triangular matrix $U \in \mathbb{R}^{3\times 3}$ is the product of the diagonal elements. In other words, prove that

$$\det(U) = \det\left(\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}\right) = u_{11}\, u_{22}\, u_{33}.$$

10. Use the formula for the determinant of a 2×2 matrix to to prove that the determinant of an lower-triangular matrix $L \in \mathbb{R}^{2\times 2}$ is the product of the diagonal elements. In other words, prove that

$$\det(L) = \det\left(\begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix}\right) = \ell_{11}\, \ell_{22}.$$

11. Use the formula for the determinant of a 3×3 matrix to to prove that the determinant of an upper-triangular matrix $U \in \mathbb{R}^{3\times 3}$ is the product of the diagonal elements. In other words, prove that

$$\det(U) = \det\left(\begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}\right) = \ell_{11}\, \ell_{22}\, \ell_{33}.$$

12. Use a determinant to find out if a given set of vectors are linearly independent. (See the invertible matrix theorem).

13. Show that if A is invertible, then $\det(A) = \dfrac{1}{\det(A)}$.

14. Find $\det(cA)$, where $c \in \mathbb{R}$ and $A \in \mathbb{R}^{n\times n}$.

Lesson 18: Determinants- Challenge Problems

1. Suppose we collect three data points $\{(x_i, y_i)\}_{i=1}^{3}$. Consider the Vandermonde matrix associated with this data:

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

Use row operations to show that $\det(A) = (x_2 - x_1)\cdot(x_3 - x_1)\cdot(x_3 - x_2)$

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