Answers **Lesson 15 Warm Up Quiz**

Math 2B: Applied Linear Algebra

True/False

For the problems below, circle T if the answer is true and circle F is the answer is false.

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1. **T** F Backward substitution is used to solve lower-triangular, linear systems.

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2. T **F** The diagonal elements of upper triangular U from the LU factorization of A are unique.

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3. **T** F Let $A = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{bmatrix}$. Notice that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Let $\mathbf{x} \in \mathbb{R}^3$ be the vector such that $A\mathbf{x} = \begin{bmatrix} 6 \\ 24 \\ 70 \end{bmatrix}$. Then $x_1 + x_2 + x_3 = 3$.

Multiple Choice

For the problems below, circle the correct response for each question.

1. Suppose we are given a matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 5 & -2 \\ 2 & -2 & 0 \end{bmatrix}$.

   Find the matrix $L \in \mathbb{R}^{3\times 3}$ from the LU factorization of A.

   A. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$ E. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

2. The LU Factorization of a given 3×3 matrix is $A = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.5 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4.5 \end{bmatrix}}_{U}$.

   Use the LU factorization of A combined with forward and backward substitution to find the solution to the linear systems problem

   $$\begin{bmatrix} 2 & 6 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -9 \end{bmatrix}$$

   Which of the following gives $-x_1 + x_2 + x_3$?

   A. -2 B. -1 C. 0 **D. 1** E. 2

3. Below we are given the LU Factorization of the matrix A and vector $\mathbf{b}$:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 6 & -4 & 2 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

Use this LU Factorization to solve the linear system $A\mathbf{x} = \mathbf{b}$ by solving the two linear systems

$$L\mathbf{y} = \mathbf{b} \qquad \text{and} \qquad U\mathbf{x} = \mathbf{y}$$

Then find $\mathbf{y}^T\mathbf{x}$:

A. -22 B. 22 C. -16 D. -18 E. 18

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4. Consider the following matrix equation

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 1 \end{bmatrix}}_{L_3} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}}_{L_2} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}}_{L_1} \cdot \underbrace{\begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 3 & 1 \\ 3 & 0 & 2 & 2 \\ 1 & 4 & 0 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & -35 \end{bmatrix}}_{U}$$

Find the lower-triangular matrix $L \in \mathbb{R}^{4\times 4}$ from the LU factorization of the matrix A:

A. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 1 & -3 & 5 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & -3 & 1 & 0 \\ -1 & 3 & -5 & 1 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ -22 & 18 & -5 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 9 & 3 & 1 & 0 \\ 40 & 12 & 5 & 1 \end{bmatrix}$ E. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 2 & -3 & 5 & -35 \end{bmatrix}$

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5. Consider the 3×3 matrices given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

As discussed in class, we can multiply the matrix A by a sequence of three elementary matrices E_1, E_2, E_3 to produce the upper-triangular matrix $U \in \mathbb{R}^{3\times 3}$ with

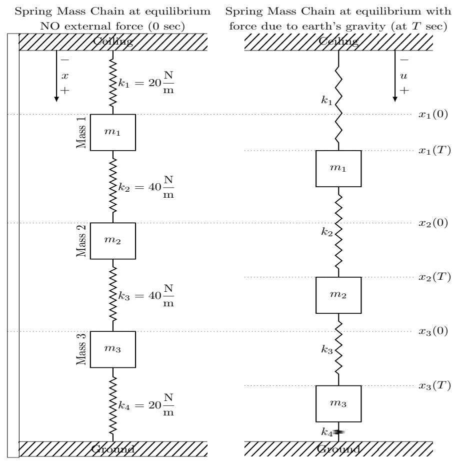
$$E_3 \cdot E_2 \cdot E_1 \cdot A = U.$$

Which of the following matrices is NOT one of the elementary matrices E_i we used to accomplish this transformation?

A. $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -9 & 0 & 1 \end{bmatrix}$ **C.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1.5 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

Free Response

For problems 1 - 2 below, consider the following mass-spring chain:



1. Recall from our previous discussion, this system results in a linear systems problem

$$\underbrace{\begin{bmatrix} 60 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 60 \end{bmatrix}}_{K} \cdot \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\mathbf{u}} = 9.8 \underbrace{\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}}_{\mathbf{m}}$$

Find the LU Factorization of matrix K by multiplying K on the left by a sequence of elementary matrices.

Solution: In order to create the LU Factorization of the matrix K, we will multiply on the left by a sequence of unit lower-triangular matrices to create an upper-triangular matrix U.

STEP 1: Identify pivot 1 and zero out all elements below this entry in pivot column 1

Since entry k_{11} of our matrix K is nonzero, we call this entry the first pivot of our matrix. We now use this pivot to eliminate all entries in column 1.

In our case, we notice that $k_{11} = 60$ is nonzero. This entry, which we've circled in our coefficient matrix below,is our first pivot. Now, we multiply this matrix on the left by L_1 to yield

$$L_1 \cdot K = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 60 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 60 \end{bmatrix} = \begin{bmatrix} 60 & -40 & 0 \\ 0 & \frac{160}{3} & -40 \\ 0 & -40 & 60 \end{bmatrix}$$

In this case, we added $c = -\frac{-40}{60}$ times row 1 from row 2. Since entry $(3, 1)$ was already zero, we need not work on that entry. The next step is to move on to column 2.

STEP 2: Identify pivot 2 and zero out all elements below this entry in pivot column 2

Since entry $(2,2)$ of our transformed matrix is nonzero, we call this entry the second pivot of our matrix. We now use this pivot to eliminate all entries below this pivot in column 2.

In the second step of our Guassian elimination process, we focus on zeroing out all strictly lower-triangular entries in column 2 below the second pivot. To do so, we use a similar unit lower-triangular matrix:

$$L_2 \cdot L_1 \cdot K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{3}{4} & 1 \end{bmatrix} \cdot \begin{bmatrix} 60 & -40 & 0 \\ 0 & \frac{160}{3} & -40 \\ 0 & -40 & 60 \end{bmatrix} = \begin{bmatrix} 60 & -40 & 0 \\ 0 & \frac{160}{3} & -40 \\ 0 & 0 & 30 \end{bmatrix} = U$$

Now, to create the full LU factorization of the matrix K, we compute the product $L = L_1^{-1} \cdot L_2^{-1}$. However, as we noted in our discussion of inverses, the inverse of L_1 is the exact same matrix except the lower triangular entries include a negative sign. In this case,

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The same pattern holds in order to calculate L_2^{-1}

$$L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{3}{4} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{3}{4} & 1 \end{bmatrix}$$

Multiplying these inverses together yields the matrix

$$L = L_1^{-1} \cdot L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{3}{4} & 1 \end{bmatrix}$$

With this, we have the lower-triangular factor L and the upper-triangular factor U of the LU factorization of K and we write

$$\underbrace{\begin{bmatrix} 60 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 60 \end{bmatrix}}_{K} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{3}{4} & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 60 & -40 & 0 \\ 0 & \frac{160}{3} & -40 \\ 0 & 0 & 30 \end{bmatrix}}_{U}$$

Since our original matrix $K \in \mathbb{R}^{3\times3}$ had only two columns with strictly lower triangular entries k_{ij} with $i > j$, we only needed to complete two simplification steps in our transformation process. Now that we have our LU factorization of K, we can solve any linear-system problem involving the coefficient matrix K using combination forward then backward substitution algorithms, as illustrated below.

2. Use the LU factorization method we have discussed in class to find the displacement vector $\mathbf{u} \in \mathbb{R}^3$ measured in meters at $t = T$ when the system is at equilibrium under the force of gravity on earth in each of the following cases:

A. $\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0.200 \\ 0.400 \\ 0.200 \end{bmatrix}$ B. $\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.0 \\ 0.5 \end{bmatrix}$

Solution: Let's begin by solving part A. The LU factorization we found above enables us to transform the nonsingular linear system problem $K \cdot \mathbf{u} = \mathbf{f}_e$ into a sequence of two related linear-systems problems

$$L \cdot \mathbf{y} = \mathbf{f}_e, \qquad U \cdot \mathbf{u} = \mathbf{y}.$$

We solve the first problem using forward substitution:

$$L \cdot \mathbf{y} = \mathbf{f}_e \quad \Longrightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1.96 \\ 3.92 \\ 1.92 \end{bmatrix}$$

$$\Longrightarrow \quad y_1 = \frac{1.96}{1} = 1.96$$

$$y_2 = \frac{3.92 - (-\frac{2}{3}) \cdot y_1}{1} = \frac{392}{75} \approx 5.8867$$

$$y_3 = \frac{1.96 - (0) \cdot y_1 - (-\frac{3}{4}) \cdot y_2}{1} = 5.88$$

Now that we have the entry-by-entry definition of $\mathbf{y}$, we can solve our second linear-systems problem using backward substitution

$$U \cdot \mathbf{u} = \mathbf{y} \quad \Longrightarrow \quad \begin{bmatrix} 60 & -40 & 0 \\ 0 & \frac{160}{3} & -40 \\ 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1.96 \\ \frac{392}{75} \\ 5.88 \end{bmatrix}$$

$$\Longrightarrow \quad u_3 = \frac{y_3}{30} = 0.196$$

$$u_2 = \frac{y_2 - (-40) \cdot u_3}{\frac{160}{3}} = 0.245$$

$$u_1 = \frac{y_1 - (-40) \cdot u_2 - (0) \cdot u_3}{60} = 0.196$$

Thus, we conclude that the displacement vector associated with this linear-systems problem is

$$\mathbf{u} = \begin{bmatrix} 0.196 & 0.245 & 0.196 \end{bmatrix}^T$$

Solution: Let's continue by solving part B. Another beautiful feature of the LU factorization is that we can use this structure to solve many linear-systems problems that depend on the same coefficient matrix quickly. In this case, we have the same sequence of two related linear-systems problems

$$L \cdot \mathbf{y} = \mathbf{f}_e, \qquad U \cdot \mathbf{u} = \mathbf{y}.$$

but this time we use a different external force vector $\mathbf{f}_e$. Again, we solve the first problem using forward substitution:

$$L \cdot \mathbf{y} = \mathbf{f}_e \implies \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4.9 \\ 9.8 \\ 4.9 \end{bmatrix}$$

$$\implies \quad y_1 = \frac{4.9}{1} = 4.9$$

$$y_2 = \frac{9.8 - (-\frac{2}{3}) \cdot y_1}{1} = \frac{196}{15} \approx 13.067$$

$$y_3 = \frac{4.9 - (0) \cdot y_1 - (-\frac{3}{4}) \cdot y_2}{1} = 14.7$$

Now that we have the entry-by-entry definition of $\mathbf{y}$, we can solve our second linear-systems problem using backward substitution

$$U \cdot \mathbf{u} = \mathbf{y} \implies \begin{bmatrix} 60 & -40 & 0 \\ 0 & \frac{160}{3} & -40 \\ 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 4.9 \\ \frac{196}{15} \\ 14.7 \end{bmatrix}$$

$$\implies \quad u_3 = \frac{y_3}{30} = 0.49$$

$$u_2 = \frac{y_2 - (-40) \cdot u_3}{\frac{160}{3}} = 0.6125$$

$$u_1 = \frac{y_1 - (-40) \cdot u_2 - (0) \cdot u_3}{60} = 0.49$$

Thus, we conclude that the displacement vector associated with this linear-systems problem is

$$\mathbf{u} = \begin{bmatrix} 0.49 & 0.6125 & 0.49 \end{bmatrix}^T$$

For problems 3 - 4, consider the nonsingular linear-systems problem

$$\underbrace{\begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ -2 \\ 8 \end{bmatrix}}_{\mathbf{b}}$$

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3. Calculate the LU factorization of A from problem 1 above.

Solution: We want to find the LU Factorization the coefficient matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

However, when we start this process, we notice that the coefficient in entry $(1, 1)$ is zero. Thus, we can not identify this as pivot 1. In order to remedy this situation, we first apply the permutation matrix:

$$P_{13} \cdot A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 6 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Now, we find the LU Factorization of this permuted matrix $P_{13} \cdot A$. We can use our work above to see

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.5 & 1 \end{bmatrix}}_{L_2} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1} \cdot \underbrace{\begin{bmatrix} 1 & 1 & 4 \\ 2 & 6 & 1 \\ 0 & 2 & 1 \end{bmatrix}}_{P_{13} \cdot A} = \underbrace{\begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & -7 \\ 0 & 0 & 4.5 \end{bmatrix}}_{U}$$

Thus, we can form the L factor by creating the matrix

$$L = L_1^{-1} \cdot L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}$$

Thus, although we could not generate an LU Factorization of A because of the bad behavior of coefficients in a pivot position, we used a permutation matrix to create a better scenario. In particular, we calculated the LU factorization of the permuted matrix

$$P_{13} \cdot A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 6 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0.5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & -7 \\ 0 & 0 & 4.5 \end{bmatrix} = L \cdot U$$

4. Solve the original linear-systems problem above using the LU Factorization you found in problem 3.

Solution: As noted above, we could not find the LU factorization of A. However, by preprocessing A to permute the rows, we found the LU factorization of a related matrix. Using this insight, we can solve our original linear-systems problem by multiplying both sides of the equation by matrix P_{13}:

$$P_{13} \cdot A \cdot \mathbf{x} = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 6 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix} = \hat{\mathbf{b}} = P_{13} \cdot \mathbf{b}$$

In this restated equation, we swapped rows 1 and 3 on BOTH sides of the equation. Now, this equivalent linear-systems problem can be solved using the LU Factorization method. First, we find the solution to the lower-triangular system:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix}}_{P_{13} \cdot \mathbf{b}}.$$

Using forward substitution, we find that

$$y_1 = \frac{8}{1} = 8$$

$$y_2 = \frac{-2 - (2) \cdot y_1}{1} = -18$$

$$y_3 = \frac{0 - (0) \cdot y_1 - (0.5) \cdot y_2}{1} = 9$$

Now, we use the solution to this first system as the right-hand side vector to the second linear system

$$\underbrace{\begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & -7 \\ 0 & 0 & 4.5 \end{bmatrix}}_{U} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 8 \\ -18 \\ 9 \end{bmatrix}}_{\mathbf{y}}.$$

This time, we solve using backward substitution to find

$$x_3 = \frac{y_3}{4.5} = 2$$

$$x_2 = \frac{y_2 - (-7) \cdot x_3}{4} = -1$$

$$x_1 = \frac{y_1 - (1) \cdot x_2 - (4) \cdot x_3}{1} = 1$$

Thus, we see that the unique solution to our original linear-systems problem is given by

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$