5.4 The Invertible Matrix Theorem

Theorem 27: The Invertible Matrix Theorem: Part 1

Let $A \in \mathbb{R}^{n\times n}$ be a square matrix with real entries. Then, the following statements are equivalent. That is, for a given matrix A, the statements are either all true or all false.

1. There is a matrix $C \in \mathbb{R}^{n\times n}$ such that $CA = I_n$.
2. There is a matrix $D \in \mathbb{R}^{n\times n}$ such that $AD = I_n$.
3. A is an invertible matrix (A is nonsingular).
4. A is row equivalent to an upper triangular matrix with nonzero entries on the main diagonal.
5. A has n pivot positions.
6. The matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = 0$.
7. The columns of A are linearly independent. In other words
$$\{A(:,1), A(:,2), ..., A(:,n)\}$$
is a linearly independent set of vectors.
8. The linear transformation $f(\mathbf{x}) = A\mathbf{x}$ is one-to-one.
9. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solutions for all $\mathbf{b} \in \mathbb{R}^n$.
10. The columns of A span $\mathbb{R}^n$.
11. The linear transformation $f(\mathbf{x}) = A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
12. The matrix A^T is invertible.

Proof. Provide a full and thorough proof of all equivalencies from "easiest" to most difficult. □

Theorem 28: The Invertible Matrix Theorem: Part 2

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix with real entries. Then, the following statements are equivalent. That is, for a given matrix A, the statements are either all true or all false.

13. The determinant of A is not zero: set $(A) \neq 0$.
14. The columns of A form a basis for $\mathbb{R}^n$
15. The column space of A is $\mathbb{R}^n$: $\text{Col}(A) = \mathbb{R}^n$.
16. The dimension of the column space of A is n: $\dim(\text{Col}(A)) = n$
17. The rank of A is n: $\text{rank}(A) = n$.
18. The null space of A is $\{\mathbf{0}\}$: $\text{Null}(A) = \{\mathbf{0}\}$.
19. The dimension of the null space of A is 0: $\dim(\text{Nulll}(A)) = 0$.
20. The orthogonal complement of the column space of A is $\{\mathbf{0}\}$. We can write this as
$$(\text{Col}(A))^{\perp} = \{\mathbf{0}\}.$$
21. The orthogonal complement of the null space of A is $\mathbb{R}^n$: We write this
$$(\text{Null}(A))^{\perp} = \mathbb{R}^n.$$
22. The row space of A is $\mathbb{R}^n$: $\text{Row}(A) = \mathbb{R}^n$.

Theorem 29: The Invertible Matrix Theorem: Part 3

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix with real entries. Then, the following statements are equivalent. That is, for a given matrix A, the statements are either all true or all false.

23. The number 0 is not an eigenvalue of A.
24. The matrix A has n non-zero singular values.