**Answers** **Lesson 13 Warm Up Quiz**

**Math 2B: Applied Linear Algebra**

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## True/False

For the problems below, circle T if the answer is true and circle F is the answer is false.

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|---|---|---|---|
| 1. | T | **F** | For square $A, B \in \mathbb{R}^{n\times n}$, if $AB = BA$ and if $A$ is invertible, then $A^{-1}B = BA^{-1}$. |
| 2. | **T** | F | If $A, B \in \mathbb{R}^{n\times n}$ invertible, then the product $AB \in \mathbb{R}^{n\times n}$ is also invertible. |
| 3. | T | **F** | If $A$ and $B$ are square and invertible, then $AB$ is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$. |
| 4. | T | **F** | If $A \in \mathbb{R}^{n\times n}$ is singular, then the columns of $A$ form a basis for $\mathbb{R}^n$. |
| 5. | T | **F** | If $A, B \in \mathbb{R}^{n\times n}$ invertible, then the sum $A + B \in \mathbb{R}^{n\times n}$ is also invertible. |
| 6. | T | **F** | The transpose of a square $n \times n$ shear matrix $S_{ij}(c)$ is the inverse of that matrix. In other words $S_{ji}(c) = (S_{ij}(c))^{-1}$. |
| 7. | T | **F** | Any square matrix $A \in \mathbb{R}^{n\times n}$ with nonzero diagonals is invertible |
| 8. | T | **F** | If $A$ is invertible and $c \neq 0$ is a real number, then $(cA)^{-1} = cA^{-1}$. |
| 9. | T | **F** | Let $A \in \mathbb{R}^{n\times n}$. If $\{E_j\}_{j=1}^k \subseteq \mathbb{R}^{n\times n}$ is a set of elementary matrices such that $$E_kE_{k-1}\cdots E_2E_1A = I_n,$$ then $A^{-1} = E_1^{-1}E_2^{-1}\cdots E_{k-1}^{-1}E_k$. |
| 10. | T | **F** | Let $n \in \mathbb{N}$ and suppost $i, j, k \in \{1, 2, ..., n\}$ with $i \neq j$. Let $P_{ij} \in \mathbb{R}^{n\times n}$ be the permutation matrix that transposes rows $i$ and $j$. Then $$\left(P_{ij} \cdot S_{jk}(c)\right)^{-1} = S_{kj}(-c).$$ |

| | | | |
|---|---|---|---|
| 11. | T | **F** | If a square matrix $A \in \mathbb{R}^{n\times n}$ has a zero on its main diagonal, then it is singular. |
| 12. | **T** | F | If $A$ is a $3 \times 3$ matrix and the equation $A\mathbf{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ has a unique solution then $A$ is invertible. |
| 13. | **T** | F | Let $n \in \mathbb{N}$ and suppose $i, k \in \{1, 2, ..., n\}$ with $i \neq k$. Then $$\left((S_{ik}(c))^T\right)^{-1} = S_{ki}(-c).$$ |
| 14. | **T** | F | For square $A, B \in \mathbb{R}^{n\times n}$, if $AB = BA$ and if $A$ is invertible, then $AB^{-1} = B^{-1}A$. |
| 15. | **T** | F | If $A, B \in \mathbb{R}^{n\times n}$ invertible, then the product $AB \in \mathbb{R}^{n\times n}$ is also invertible. |
| 16. | T | **F** | If $A \in \mathbb{R}^{3\times 3}$ has three pivot columns, then it is possible to find invertible matrices $E_1, E_2, ..., E_p \in \mathbb{R}^{3\times 3}$ such that $$E_pE_{p-1}\cdots E_2E_1A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$ |

## Multiple Choice

For the problems below, circle the correct response for each question.

1. Let $M = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 2 \end{bmatrix}$ Then $M^{-1}$ is given by which of the following:

A. $\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$

**B.** $\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & -0.5 & 0.5 \end{bmatrix}$

C. $\begin{bmatrix} -0.5 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0 \\ 0 & -0.5 & -0.5 & 0 \\ 0 & 0 & -0.5 & -0.5 \end{bmatrix}$

D. $\begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & 0 & 0.5 & -0.5 \end{bmatrix}$

E. None of these.

2. Let $M = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$ Then $M^{-1}$ is given by which of the following:

A. $M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 5 & 1 \end{bmatrix}$

B. $M^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ -2 & 5 & -1 \end{bmatrix}$

C. $M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix}$

**D.** $M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 5 & 1 \end{bmatrix}$**.**

E. None of these.

3. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{bmatrix}$. If $B = A^{-1}$, which of the following gives $B(1,:)$?

A. $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$ **B.** $\begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$ C. $\begin{bmatrix} 2 & -2 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$ E. $\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$

4. Let $B = \overbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&7&1\end{bmatrix}}^{E_3} \cdot \overbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&-5&0&1\end{bmatrix}}^{E_2} \cdot \overbrace{\begin{bmatrix}1&0&0&0\\3&1&0&0\\-2&0&1&0\\1&0&0&1\end{bmatrix}}^{E_1}$ Find $B^{-1}$:

A. $\begin{bmatrix}1&0&0&0\\3&1&0&0\\-2&1&1&0\\1&-5&7&1\end{bmatrix}$

**B.** $\begin{bmatrix}1&0&0&0\\-3&1&0&0\\2&-1&1&0\\-1&5&-7&1\end{bmatrix}$

C. $\begin{bmatrix}1&-3&2&-1\\0&1&-1&5\\0&0&1&-7\\0&0&0&1\end{bmatrix}$

D. $E_3^{-1} \cdot E_2^{-1} \cdot E_1^{-1}$

E. $\begin{bmatrix}1&3&-2&1\\0&1&1&-5\\0&0&1&7\\0&0&0&1\end{bmatrix}$

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5. Let $M = \begin{bmatrix}1&-1&2\\0&1&-5\\0&0&1\end{bmatrix}$. Then $M^{-1}$ is given by which of the following:

A. $\begin{bmatrix}1&1&-2\\0&1&5\\0&0&1\end{bmatrix}$ B. $\begin{bmatrix}-1&1&-2\\0&-1&5\\0&0&-1\end{bmatrix}$ C. $\begin{bmatrix}1&1&-3\\0&1&5\\0&0&1\end{bmatrix}$ **D.** $\begin{bmatrix}1&1&3\\0&1&5\\0&0&1\end{bmatrix}$ E. $\begin{bmatrix}1&-1&-2\\0&1&-5\\0&0&1\end{bmatrix}$

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6. Let $A = \begin{bmatrix}2&1&1\\4&5&2\\2&-2&0\end{bmatrix}$. If we let $\mathbf{x}$ be the solution $\mathbf{x}$ to the linear system

$$A\mathbf{x} = \begin{bmatrix}1\\2\\2\end{bmatrix}$$

and calculate $c = \begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix} \begin{bmatrix}1\\-1\\1\end{bmatrix}$, then the constant $c$ is given by

A. $c = 1$ B. $c = -1$ **C.** $c = 0$ D. $c = 2$ E. None of these.

7. Let $B = A^{-1}$ where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

Then, which of the following gives $\Big(B(1,:)\Big)^T$?

A. $A^{-1}$ does not exist    B. $\begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$    C. $\begin{bmatrix} 3 & 4 & -5 \end{bmatrix}$    **D.** $\begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$    E. $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

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8. Consider the $3 \times 3$ matrix $A$ from the problem above. Suppose we use this matrix in the following linear-systems problem

$$\begin{bmatrix} 3 & 1 & -2 \\ -3 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

If $\mathbf{x}$ is the solution to this linear-system problem, then which of the following gives the value of

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} ?$$

A. $-3$    **B.** $-2$    C. $-1$    D. $0$    E. $2$

## Free Response

1. Let $\mathbf{w} \in \mathbb{R}^n$ be a vector such that $\mathbf{w}^T\mathbf{w} = 1$. The $n \times n$ matrix

$$H = I_n - 2\mathbf{w}\mathbf{w}^T.$$

is called a **Householder matrix**.

   A. Show that $H = H^T$ (in other words, show that $H$ is symmetric).

   B. How that $H^{-1} = H^T$.

2. Let $n, i, k \in \mathbb{N}$ such that $1 \le i \le n$, $1 \le k \le n$ and $i \ne k$. Suppose that $c \in \mathbb{R}$.

(a) Show $\Big(S_{ik}(c)\Big)^{-1} = S_{ik}\,(-c)$

**Solution:** Suppose $i, k, n \in \mathbb{N}$ such that $1 \le i, k \le n$ with $i \ne k$ and suppose $c \in \mathbb{R}$ is a nonzero constant. Let's begin by proving that $\Big(S_{ik}(c)\Big)^{-1} = S_{ik}(-c)$ by multiplying $S_{ik}(c)$ by the stated inverse to produce $I_n$. Consider

$$
\begin{aligned}
S_{ik}(c)\cdot S_{ik}(-c) &= \left(I_n + c\,\mathbf{e}_i\,\mathbf{e}_k^T\right)\cdot\left(I_n - c\,\mathbf{e}_i\,\mathbf{e}_k^T\right)\\
&= I_n - c\,\mathbf{e}_i\,\mathbf{e}_k^T + c\,\mathbf{e}_i\,\mathbf{e}_k^T\cdot I_n - c^2\,\left(\mathbf{e}_i\,\mathbf{e}_k^T\right)\cdot\left(\mathbf{e}_i\,\mathbf{e}_k^T\right)\\
&= I_n - c\,\mathbf{e}_i\,\mathbf{e}_k^T + c\,\mathbf{e}_i\,\mathbf{e}_k^T - c^2\,\mathbf{e}_i\,\left(\mathbf{e}_k^T\,\mathbf{e}_i\right)\,\mathbf{e}_k^T
\end{aligned}
$$

Notice that the matrix-matrix product $\mathbf{e}_k^T\,\mathbf{e}_i$ results in a scalar output equivalent to the inner product $\mathbf{e}_k\cdot\mathbf{e}_i$. If $j \in \mathbb{N}$ with $1 \le j \le n$ we know $\mathbf{e}_j \in \mathbb{R}^n$ is the $j$th elementary basis vector with all zero entries except the $j$th coefficient, which has value equal to one. Because of this structure and since $i \ne k$ by assumption, we see that $\mathbf{e}_k^T\,\mathbf{e}_i = \mathbf{e}_k\cdot\mathbf{e}_i = 0$. With this we have,

$$
S_{ik}(c)\cdot S_{ik}(-c) = I_n - c\,\mathbf{e}_i\,\mathbf{e}_k^T + c\,\mathbf{e}_i\,\mathbf{e}_k^T = I_n
$$

We conclude that $\Big(S_{ik}(c)\Big)^{-1} = S_{ik}(-c)$.

(b) Show $\Big(D_i(c)\Big)^{-1} = D_i\left(\frac{1}{c}\right)$

**Solution:** Let's establish that $\Big(D_i(c)\Big)^{-1} = D_i\left(\frac{1}{c}\right)$. To this end, consider

$$
\begin{aligned}
D_i(c)\cdot D_i\,(1/c) &= \left(I_n + (c-1)\,\mathbf{e}_i\,\mathbf{e}_i^T\right)\cdot\left(I_n + (1/c-1)\,\mathbf{e}_i\,\mathbf{e}_i^T\right)\\
&= I_n\cdot\left(I_n + (1/c-1)\,\mathbf{e}_i\,\mathbf{e}_i^T\right) + (c-1)\,\mathbf{e}_i\,\mathbf{e}_i^T\cdot\left(I_n + (1/c-1)\,\mathbf{e}_i\,\mathbf{e}_i^T\right)\\
&= I_n + (1/c-1)\,\mathbf{e}_i\,\mathbf{e}_i^T + (c-1)\,\mathbf{e}_i\,\mathbf{e}_i^T + (c-1)\,(1/c-1)\,\mathbf{e}_i\,\mathbf{e}_i^T\,\mathbf{e}_i\,\mathbf{e}_i^T\\
&= I_n + (1/c+c-2)\,\mathbf{e}_i\,\mathbf{e}_i^T + (c-1)\,(1/c-1)\,\mathbf{e}_i\,\mathbf{e}_i^T
\end{aligned}
$$

Using distributivity of scalar multiplication over addition, we see

$$
(c-1)\,(1/c-1) = 2 - c - 1/c.
$$

Thus, we have

$$
D_i(c)\cdot D_i\,(1/c) = I_n + (1/c+c-2)\,\mathbf{e}_i\,\mathbf{e}_i^T + (2-c-1/c)\,\mathbf{e}_i\,\mathbf{e}_i^T = I_n
$$

By definition of the matrix inverse, we have $\Big(D_i(c)\Big)^{-1} = D_i\left(\frac{1}{c}\right)$

3. Recall that Cramer's formula for the inverse of a $2 \times 2$ matrix $A \in \mathbb{R}^2$ is given by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

where $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.

(a) Using a sequence of elementary matrices, transform $A$ into $I_2$. Show each matrix you use.

**Solution:** Suppose $A \in \mathbb{R}^{2\times 2}$ is nonsingular. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

In order to produce the inverse of $A$, let's transform $A$ into $I_2$ using elementary matrices. To this end, let $\delta = a_{11}a_{22} - a_{21}a_{12}$. As we will see in Chapter 7, we call $\delta$ the determinant of $A$ and $\delta \neq 0$ if and only if $A$ is invertible. As we will see in the next section, since $A$ is invertible, we know that either $a_{11} \neq 0$ or $a_{21} \neq 0$. Thus, let's assume that $a_{11} \neq 0$. Now, we reduce $A$ to $I_2$. Consider the step-by-step calculation beginning with the product

$$S_{21}\left(-a_{21}/a_{11}\right) \cdot A = \begin{bmatrix} 1 & 0 \\ -a_{21}/a_{11} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & \delta/a_{11} \end{bmatrix}$$

Now we consider

$$D_2\left(a_{11}/\delta\right) \cdot D_1\left(1/a_{11}\right) = \begin{bmatrix} 1/a_{11} & 0 \\ 0 & a_{11}/\delta \end{bmatrix}$$

We can multiply our product by this diagonal matrix to find

$$\begin{bmatrix} 1/a_{11} & 0 \\ 0 & a_{11}/\delta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ 0 & \delta/a_{11} \end{bmatrix} = \begin{bmatrix} 1 & a_{12}/a_{11} \\ 0 & 1 \end{bmatrix}$$

Finally, multiplying this entire product on the left-hand side by we see

$$\begin{bmatrix} 1 & -a_{12}/a_{11} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a_{12}/a_{11} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Write $A^{-1}$ as a product of the elementary matrices and confirm Cramer's Rule.

**Solution:** From our work in part (a) above, we see

$$A^{-1} = S_{12}\left(-a_{12}/a_{11}\right) \cdot D_2\left(a_{11}/\delta\right) \cdot D_1\left(1/a_{11}\right) \cdot S_{21}\left(-a_{21}/a_{11}\right).$$

The reduction of matrix $A$ into $I_2$ above gives a constructive mechanism to explicitly calculate $A^{-1}$. We begin by finding the product

$$\Big(D_2\left(a_{11}/\delta\right) \cdot D_1\left(1/a_{11}\right)\Big) \cdot S_{21}\left(-a_{21}/a_{11}\right)$$

given as

$$\begin{bmatrix} 1/a_{11} & 0 \\ 0 & a_{11}/\delta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a_{21}/a_{11} & 1 \end{bmatrix} = \begin{bmatrix} 1/a_{11} & 0 \\ -a_{21}/\delta & a_{11}/\delta \end{bmatrix}$$

We can calculate $A^{-1}$ using the product

$$\begin{bmatrix} 1 & -\dfrac{a_{12}}{a_{11}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{a_{11}} & 0 \\ -\dfrac{a_{21}}{\delta} & \dfrac{a_{11}}{\delta} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a_{11}} + \dfrac{a_{21}a_{12}}{a_{11}\delta} & -\dfrac{a_{12}}{a_{11}} \\ -\dfrac{a_{21}}{\delta} & \dfrac{a_{11}}{\delta} \end{bmatrix} = \frac{1}{\delta} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

This is exactly what we wanted to show.