Chapter 5

The Nonsingular Linear-Systems Problem

5.1 Linear Systems of Equations

Definition 5.1: The Square Linear-Systems Problem

Let $n \in \mathbb{N}$. Let $A \in \mathbb{R}^{n \times n}$ be a given square, nonsingular matrix and $\mathbf{b} \in \mathbb{R}^n$ be a given vector. Then the square linear-systems problem is to find an unknown vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$A \cdot \mathbf{x} = \mathbf{b}.$$

Let's define the function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mathbf{x}) = A \cdot \mathbf{x}$. Notice that we have the following:

$$\begin{aligned} \text{Domain}(f) &= \mathbb{R}^n \\ \text{Codomain}(f) &= \mathbb{R}^n \\ \text{Rng}(f) &= \{\mathbf{b} \in \mathbb{R}^n \,:\, \mathbf{b} = A \cdot \mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}. \end{aligned}$$

Remember from our discussion of the matrix-vector multiplication, an equivalent description of the range of this function f was given by

$$\text{Rng}(f) = \text{ Span } \{A(:,k)\}_{k=1}^{n}.$$

In other words, we say that the $\mathbf{b}$ is in the range of f if and only if we can write $\mathbf{b}$ as a linear combination of the columns of the matrix A.

When studying the square linear-systems problem, we need to create our square matrix A and a vector $\mathbf{b}$. We then need to calculate all possible vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ or to conclude that no such $\mathbf{x}$ exists. From this standpoint, we can view the linear-systems problem as the inverse of the matrix-vector multiplication problem.

In this section, we will explore the various aspects of this problem, discover a number of applications that give rise to the square linear-systems problem and begin to create theory to help us generate a solution to such systems.

We begin our study of techniques to solve square linear systems with a discussion of problems that contain very special, structured matrices.

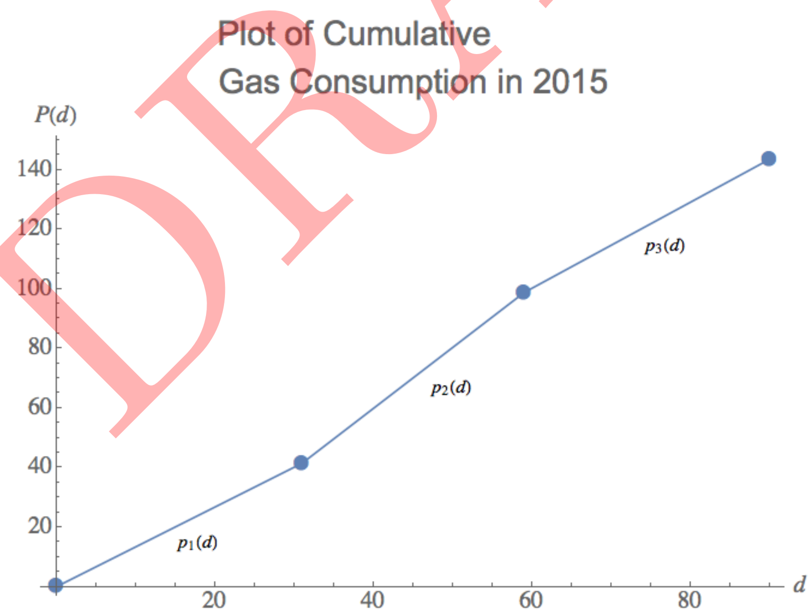
EXAMPLE 5.1.1

Let's create a piecewise linear graph to model our gas consumption in 2015. Below is a data set consisting of the author's gasoline consumption from January 1, 2015 at 12am to December 31, 2015 at 11:59pm.

Data point i in 2015 (by Number)	Day number d_i at end of month	Cumulative volume of of gas g_i (in gallons)
1	0	0.000
2	31	41.490
3	59	98.804
4	90	143.622

The first coordinate d_i of each data point (d_i, g_i) represents the number of days of the year that have past at the beginning of month i. For example, we see that $d_1 = 0$ because zero days have past on January 1, 2015 at 12am. Next we have that $d_2 = 31$ since there are 31 that have past by on the morning of February 1, 2015 at 12:00am. Similarly, $d_3 = 59$ since on March 1, 2015 at 12:00am a total of 59 days have past since the start of the year (2015 was not a leap year). The corresponding values of g_i denote the cumulative gas consumed realized by the author at the beginning of month i.

We begin our modeling problem by graphing each data point (d_i, g_i) on a single access to compare the day of the year to the cumulative gallons consumed on that day.



Suppose want to interpolate linearly between the first three collected data points.

In other words, suppose we want to create a piecewise linear polynomial

$$P(d) = \begin{cases} p_1(d) & \text{if } d_1 \le d \le d_2 \\ p_2(d) & \text{if } d_2 \le d \le d_3 \\ p_3(d) & \text{if } d_3 \le d \le d_4 \end{cases}$$

where $P(d)$ is the cumulative gallons we expect the author to have used on day number d in the first quarter of 2015. Each segment of the piecewise linear function given by

$$p_i(d) = m_i(d - d_i) + g_i$$

for any $d_i \le d \le d_{i+1}$ where $i = 1, 2, 3$. The slopes m_i are unknown. Our model $P(d)$ should "connects the dots" using lines. In other word, to create a continuous piecewise linear function, we impose two conditions:

$$p_i(d_i) = g_i, \qquad\qquad p_i(d_{i+1}) = g_{i+1}.$$

Thus, for each i, we have

$$g_{i+1} = p_i(d_{i+1}) = m_i(d_{i+1} - d_i) + g_i$$

These conditions result in a system of 3 equations in 3 unknowns:

$$\begin{aligned} g_2 - g_1 &= m_1(d_2 - d_1) \\ g_3 - g_2 &= m_2(d_3 - d_2) \\ g_4 - g_3 &= m_3(d_4 - d_3) \end{aligned}$$

We write these linear equations in matrix form to set up a linear-systems problem

$$\begin{bmatrix} 31 & 0 & 0 \\ 0 & 28 & 0 \\ 0 & 0 & 31 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 41.490 \\ 57.314 \\ 44.818 \end{bmatrix}$$

The individual coefficients m_i of the solution vector $\mathbf{m} \in \mathbb{R}^3$ represent the average gallons used per day used during month i for $i = 1, 2, 3$. The entry m_i of the solution can also be interpreted as the slope of the line segment connecting data points (d_i, g_i) to (d_{i+1}, g_{i+1}) for $i = 1, 2, 3$. Because our coefficient matrix is diagonal and all entires on the main diagonal are nonzero, we know our linear-systems problem in this example has a unique solution.

The process outlines in the example above is a special case of a much more general modeling technique known as linear spline interpolations. Let's see how we might create a general linear piecewise polynomial to interpolate $(n+1)$ collected data points.

EXAMPLE 5.1.2

Suppose we run an experiment and collect $n+1$ data points

$$\{(t_i, y_i)\}_{i=1}^{n+1} \subseteq \mathbb{R}.$$

Let's create a model to interpolate any value between two collected points (t_i, y_i) and (t_{i+1}, y_{i+1}) for $i = 1, 2, ..., n$. We can do so by using a set of individual line segments. In other words, we can connect the dots using a piecewise linear function. Given any $t_i \leq t \leq t_{i+1}$, we define the line that connects our to points using the point-slope form of a line as follows

$$y - y_i = m_i(t - t_i)$$

where m_i is the unknown slope of this line. Based on the definition of m_i and by doing some basic arithmetic, we see we can write the equation for this line $p_i(t)$ as follows:

$$p_i(t) = m_i(t - t_i) + y_i.$$

If we try to connect the dots to create a continuous piecewise function, we will impose two conditions:

$$\begin{aligned} p_i(t_i) &= y_i \\ p_i(t_{i+1}) &= y_{i+1}. \end{aligned}$$

Thus, for each i, we have

$$y_{i+1} = p_i(t_{i+1}) = m_i(t_{i+1} - t_i) + y_i$$

These conditions result in a system of n equations in n unknowns:

$$\begin{aligned} y_2 - y_1 &= m_1(t_2 - t_1) \\ y_3 - y_2 &= m_2(t_3 - t_2) \\ &\vdots \\ y_{n+1} - y_n &= m_n(t_{n+1} - t_n). \end{aligned}$$

We can create the corresponding linear-systems problem by stating these equations in matrix form

$$\begin{bmatrix} (t_2 - t_1) & 0 & \cdots & 0 \\ 0 & (t_3 - t_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (t_{n+1} - t_n) \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} (y_2 - y_1) \\ (y_3 - y_2) \\ \vdots \\ (y_{n+1} - y_n) \end{bmatrix}$$

Now that we've state the most general form of this piecewise linear interpolation problem, let's create a specific model to bring to life one possible application.

Suppose we have a square linear-systems problem given by

$$D\mathbf{x} = \mathbf{b}$$

where the coefficient matrix $D \in \mathbb{R}^{n \times n}$ is square and diagonal and $\mathbf{b} \in \mathbb{R}^n$. Then, we can categorize each solution to this system based on the sparsity structure of D and the nonzero entries of $\mathbf{b}$. We see that our linear-systems problem has

i. No solution iff $d_{ii} = 0$ and $b_i \neq 0$ for some $i \in \{1, 2, ..., n\}$

ii. A unique solution iff $d_{ii} \neq 0$ for all $i \in \{1, 2, ..., n\}$

iii. Infinitely many solutions iff $d_{ii} = 0$ AND $b_i = 0$ for some $i \in \{1, 2, ..., n\}$

Using the structure of diagonal matrices, we can immediately classify the cardinality of the solution set. More importantly, in the case that we have a unique solution, we can calculate this solution very quickly as:

$$x_i = \frac{b_i}{d_{ii}}, \qquad\qquad \text{for all } i = 1, 2, ..., n$$

Because of the beautiful simplicity of the solution, we consider square linear-systems problems that have diagonal coefficient matrices with nonzero entires to be the gold standard of all linear-systems problems. If possible, we will work to transform all matrices into diagonal form and solve accordingly. As we will see, this theme will arise throughout our study of linear systems and beyond.

As we've seen above, square linear-systems with diagonal coefficient matrices are wonderful. However, linear-systems problems that arise from mathematical modeling rarely result in diagonal coefficient matrices. More often than not, matrix modeling results in a structured coefficient matrix A. When attempting to solve the corresponding linear-systems problem, we need to find a way to translate this matrix into a much better form. While the idea of transforming our matrix into diagonal form is appealing, these transformations often prove to be expensive in practice. Instead, let's look for the next best option, a square linear system

$$U\mathbf{x} = \mathbf{y}$$

where $U \in \mathbb{R}^{n\times n}$ is upper-triangular.

As we will see in our discussion of solution sets for linear systems and in our work with the LU Factorization of a square matrix, upper-triangular matrices play a very special role in solving linear systems. For now, let's investigate how we might solve a 5×5 system with an upper-triangular coefficient matrix U.

EXAMPLE 5.1.3

Let's look at a system of 5 linear equations in 5 unknowns that results in an upper-triangular coefficient matrix U. Suppose that the diagonal elements of U are nonzero and consider the linear system

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} \\ 0 & u_{22} & u_{23} & u_{24} & u_{25} \\ 0 & 0 & u_{33} & u_{34} & u_{35} \\ 0 & 0 & 0 & u_{44} & u_{45} \\ 0 & 0 & 0 & 0 & u_{55} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

Now, we can focus on the scalar versions of this matrix equation by looking at the individual row entries of the left- and right-hand side. We see:

$$\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 + u_{14}x_4 + u_{15}x_5 &= y_1 \\ 0 + u_{22}x_2 + u_{23}x_3 + u_{24}x_4 + u_{25}x_5 &= y_2 \\ 0 + \quad 0 + u_{33}x_3 + u_{34}x_4 + u_{35}x_5 &= y_3 \\ 0 + \quad 0 + \quad 0 + u_{44}x_4 + u_{45}x_5 &= y_4 \\ 0 + \quad 0 + \quad 0 + \quad 0 + u_{55}x_5 &= y_5 \end{aligned}$$

We see that the last equation has only one unknown. Moreover, if $u_{ii} \neq 0$ for all values of i, we solve for the unknown

$$x_5 = \frac{1}{u_{55}} \cdot (y_5)$$

We now know the value of coefficient x_5 and we have eliminated one of our unknowns. We move up to the second to last equation

$$u_{44}x_4 + u_{45}x_5 = y_4, \qquad \Longrightarrow \qquad x_4 = \frac{1}{u_{44}} \cdot (y_4 - u_{45}x_4)$$

We continue with row 3 and solve for x_3 by recognizing

$$u_{33}x_3 + u_{34}x_4 + u_{35}x_5 = y_3, \qquad \Longrightarrow \qquad x_3 = \frac{1}{u_{33}} \cdot (y_3 - u_{34}x_4 - u_{35}x_5)$$

$$\Longrightarrow \qquad x_3 = \frac{1}{u_{33}} \cdot \left(y_3 - \sum_{j=4}^{5} u_{3j}x_j \right)$$

Next, we find x_2 using the formulas

$$u_{22}x_2 + u_{23}x_3 + u_{24}x_4 + u_{25}x_5 = y_2, \quad \Longrightarrow \quad x_2 = \frac{1}{u_{22}} \cdot (y_2 - u_{23}x_3 - u_{24}x_4 - u_{25}x_5)$$

$$\Longrightarrow \qquad x_2 = \frac{1}{u_{22}} \cdot \left(y_2 - \sum_{j=3}^{5} u_{2j}x_j \right)$$

Finally, we use the pattern from our work above to solve for x_1:

$$x_1 = \frac{1}{u_{11}} \cdot \left(y_1 - \sum_{j=2}^{5} u_{2j}x_j \right)$$

We have now solved our entire linear systems problem for unknowns x_i. This process, described in general below, will be known as backward substitution.

In the example above, we describe the process of using backward substitution to solve a special case of a 5×5 linear-system with upper-triangular coefficient matrix U. We can generalize this process to solve $n \times n$ upper-triangular systems.

Theorem 20: Backward Substitution: Upper-Triangular $U\mathbf{x} = \mathbf{y}$

Let $U\mathbf{x} = \mathbf{y}$ be a given linear-systems problem with upper-triangular $U \in \mathbb{R}^{n\times n}$ and $\mathbf{y} \in \mathbb{R}^n$. If $u_{ii} \neq 0$ for all $i \in \{1, 2, ..., n\}$, then our linear-system problem has a unique solution. Further, we can find this solution using the following algorithm:

$$x_n = \frac{y_n}{u_{nn}}$$

$$x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{j=i+1}^{n} u_{ij}\, x_j\right)$$

where $i = (n-1), (n-2), ..., 2, 1$.

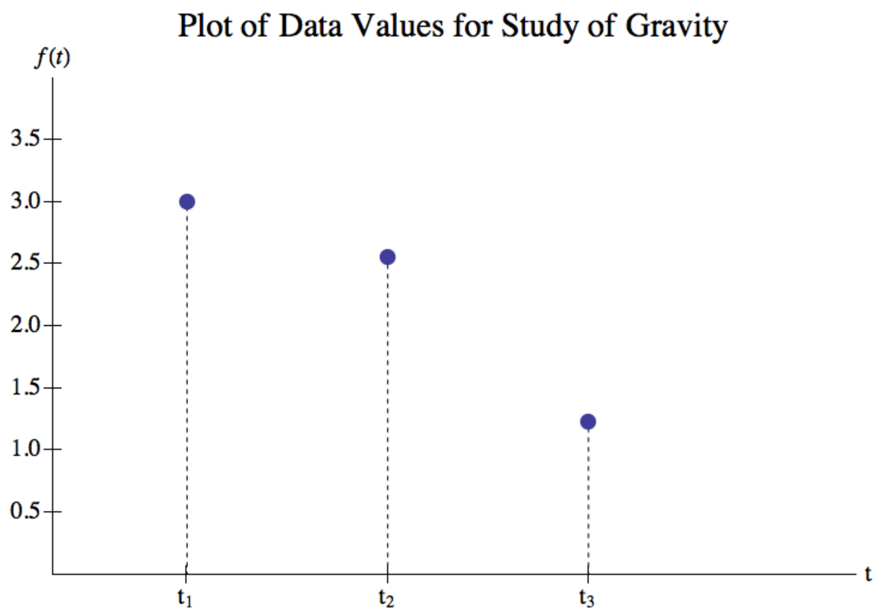
Now that we have a mechanism to solve upper-triangular systems, let's look at a modeling problem in which we can create an upper-triangular system.

EXAMPLE 5.1.4

Suppose we are working on a physics laboratory project that asks us to explore the effect of earth's gravity on an object in free fall close to the earth's surface. As part of this lab, we record the height of a falling object at three separate instances in time. The result is a three-point data set $\{(t_i, h_i)\}_{i=1}^{3}$ where h_i is the observed height of our object, measured in meters, at time t_i, measured in seconds, for $i = 1, 2, 3$. In our case, we collect the following data

Measurement Number	Time t_i at which data was collected (in seconds)	Measured height h_i in meters
1	3.0	3.000
2	3.3	2.559
3	3.6	1.236

Using this data, let's create a mathematical model for our findings. First, let's decide which family of functions we will use to describe the behavior of our data. We begin by graphing the data.



We see that this data seems non-linear (doesn't fit on a straight line). We might guess that we can use a quadratic polynomial to model our data. Also, from our study of introductory physics, we confirm that the family of quadratic functions works nicely to model the position of a free-falling object close to earth's surface. Thus, we choose the general function

$$h(t) = a_0 + a_1 t + a_2 t^2$$

to attempt to fit our data. We want to find the coefficients a_0, a_1, a_2 that most closely fit our data so that

$$h_i = h(t_i) = a_0 + a_1 t_i + a_2 t_i^2$$

With this, we can set up one linear equation for each data point we've collected

$$\begin{aligned} 3.000 &= a_0 + 3.0a_1 + 9.00a_2 \\ 2.559 &= a_0 + 3.3a_1 + 10.89a_2 \\ 1.236 &= a_0 + 3.6a_1 + 12.96a_2 \end{aligned}$$

We've set up this system of three equations and three unknowns by evaluating our ideal model $h(t)$ at each input time t_i for $i = 1, 2, 3$. We can create our coefficient matrix for this linear system and re-write this linear system of equations as a linear-systems problem using our knowledge of matrix-vector multiplication:

$$\underbrace{\begin{bmatrix} 1 & 3.0 & 9.00 \\ 1 & 3.3 & 10.89 \\ 1 & 3.6 & 12.96 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 3.000 \\ 2.559 \\ 1.236 \end{bmatrix}}_{\mathbf{b}}$$

Notice that in the current form, our stated linear-systems problem is difficult to solve. Each equation involves all three variables and it's not clear the best mechanism we should use to attempt to isolate each variable. Instead of attempting to solve our stated problem, let's transform $A\mathbf{x} = \mathbf{b}$ into the system $U\mathbf{x} = \mathbf{y}$ where U is an upper-triangular matrix. To do so, we introduce zeros to the strictly lower-triangular portion of matrix A using elementary row operations.

STEP 1: Identify the first pivot

Identify the entry with row index 1 and column index 1. Make sure this entry is nonzero and refer to this nonzero entry as the first pivot of our matrix. Refer to column 1 as the first pivot column.

In our case, we notice that $a_{11} = 1$ is nonzero. Thus, we call this entry our first pivot, which we've circled in our coefficient matrix below.

$$\begin{bmatrix} 1 & 3.0 & 9.00 \\ 1 & 3.3 & 10.89 \\ 1 & 3.6 & 12.96 \end{bmatrix}$$

Further, we call the first column $A(:,1)$ of our coefficient matrix A the first pivot column. We now sequentially alter the structure of this pivot column to simplify our system of equations.

STEP 2: Create zeros in all entries below the first pivot

Multiply the original system of equation by a sequence of shear matrices to introduce zeros in all entries in our pivot column $A(:,1)$ that are under the pivot.

In this case, we will need to zero both entries a_{21} and a_{31} in the first column.

$$\begin{bmatrix} 1 & 3.0 & 9.00 \\ 1 & 3.3 & 10.89 \\ 1 & 3.6 & 12.96 \end{bmatrix}$$

We begin by transforming a_{21} to zero. We notice that we can accomplish our desired transformation with the linear combination $-1\,A(1,:) + A(2,:)$ given by

$$-1\begin{bmatrix} 1 & 3.0 & 9.00 \end{bmatrix} + 1\begin{bmatrix} 1 & 3.3 & 10.89 \end{bmatrix} = \begin{bmatrix} 0 & 0.3 & 1.89 \end{bmatrix}$$

When changing row 2, we will not touch either rows 1 or 3. To accomplish this transformation, we multiply our matrix A on the left-hand side by the appropriate shear matrix $S_{ik}(c)$. Recall from our discussion of shear matrices in matrix-matrix multiplication that we choose $k = 1$ and $i = 2$ because we introduce a zero by adding a scalar multiple of row 1 to row 2. Moreover, we let $c = -a_{21}/a_{11} = -1$ and thus multiply A on the left by $S_{21}(-1)$ as follows

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3.0 & 9.00 \\ 1 & 3.3 & 10.89 \\ 1 & 3.6 & 12.96 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3.00 \\ 2.559 \\ 1.236 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3.0 & 9.00 \\ 0 & 0.3 & 1.89 \\ 1 & 3.6 & 12.96 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3.00 \\ -0.442 \\ 1.236 \end{bmatrix}$$

Because we are working with an equation, anything we do to the left-hand side, we must also do the right-hand side, resulting in a transformed right-hand side vector $S_{21}(-1) \cdot \mathbf{b}$.

Let's continue by creating a zero in the third row and first column given by $a_{31} = 1$. To zero out this elements, we take the linear combination $-1\,A(1,:)+A(3,:)$ given by

$$-1\begin{bmatrix}1 & 3.0 & 9.00\end{bmatrix} + 1\begin{bmatrix}1 & 3.6 & 12.96\end{bmatrix} = \begin{bmatrix}0 & 0.6 & 3.96\end{bmatrix}$$

When changing row 3, we will not touch either rows 1 or 2. Again, we realize this transformation by multiplying by shear matrix $S_{31}(-1)$. where we choose $k = 1$ and $i = 3$ because we add a scalar multiple of row 1 to row 3. We set $c = -a_{31}/a_{11} = -1$ to find

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 3.0 & 9.00\\ 0 & 0.3 & 1.89\\ 1 & 3.6 & 12.96\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}3.00\\ -0.442\\ 1.236\end{bmatrix}$$

$$\begin{bmatrix}1 & 3.0 & 9.00\\ 0 & 0.3 & 1.89\\ 0 & 0.6 & 3.96\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix} = \begin{bmatrix}3.00\\ -0.442\\ -1.764\end{bmatrix}$$

STEP 3: Identify next pivot

Move to the next row down and next column to the right. Call the first nonzero entry in this row the second pivot. The column that this pivot is in is called the second pivot column.

In the case of our transformed system, we see that the first nonzero entry in row 2 is in entry $(2, 2)$. We identify this as our second pivot and circle this pivot in the matrix below.

$$\begin{bmatrix}1 & 3.0 & 9.00\\ 0 & 0.3 & 1.89\\ 0 & 0.6 & 3.96\end{bmatrix}$$

The second column of this matrix is the pivot column corresponding to our current pivot. We will use our pivot to sequentially simplify the entries below our pivot in this pivot column.

STEP 4: Create zeros in all entries below current pivot

Multiply the current system of equations by a sequence of shear matrices to introduce zeros in all entries under the current pivot in our current pivot column.

Let's zero out the coefficients below our current pivot. In this case, we need only introduce one zero in the third row and second column.

$$\begin{bmatrix} 1 & 3.0 & 9.00 \\ 0 & 0.3 & 1.89 \\ 0 & 0.6 & 3.96 \end{bmatrix}$$

Here we multiply by the shear matrix $S_{32}(-2)$, where we choose $k = 2$ and $i = 3$ because we will introduce a zero by adding a scalar multiple of row 2 to row 3. We choose scalar $c = -0.6/0.3 = -2$ to find:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3.0 & 9.00 \\ 0 & 0.3 & 1.89 \\ 0 & 0.6 & 3.96 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3.00 \\ -0.442 \\ -1.764 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3.0 & 9.00 \\ 0 & 0.3 & 1.89 \\ 0 & 0.0 & 0.18 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3.00 \\ -0.441 \\ -0.882 \end{bmatrix}$$

Now that we have simplified our current pivot column we move onto find the next pivot column. We continue identifying pivots and introducing zeros below these pivots via multiplication with shear matrices until we create an upper-triangular matrix.

STEP 5: Repeat until all sub-diagonal elements are zero

Repeat steps 3 and 4 until our transformed linear system has an upper-triangular coefficient matrix U.

In our problem, we were lucky to have finished our reduction to upper-triangular form in three steps resulting in the transformed linear system

$$\underbrace{\begin{bmatrix} 1 & 3.0 & 9.00 \\ 0 & 0.3 & 1.89 \\ 0 & 0.0 & 0.18 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 3.00 \\ -0.441 \\ -0.882 \end{bmatrix}}_{\mathbf{y}}$$

In fact, we've tracked all three of the shear matrix multiplications needed to transform our original matrix A into our upper triangular matrix U. Setting $E = S_{32}(-2) \cdot S_{31}(-1) \cdot S_{21}(-1)$, we see that we can map our original system to our simplified system:

$$EA = U, \qquad E\mathbf{b} = \mathbf{y}$$

We've designed our transformations so any solution of our new system $U\mathbf{x} = \mathbf{y}$ will also be a solution to our original linear system of equations $A\mathbf{x} = \mathbf{b}$.

Moreover, we can solve our equivalent system $U\mathbf{x} = \mathbf{y}$ using backward substitution. Let's begin by solving for a_2 by solving the linear equation resulting from row 3 of our matrix-vector equation. We see that for our equivalent system, we have

$$0.18a_2 = -0.882 \qquad \Longrightarrow \qquad a_2 = \frac{-0.882}{0.18} = -4.9$$

Using the linear equation resulting from row 2, we have

$$0.3a_1 + 1.89a_2 = -0.441 \qquad \Longrightarrow \qquad a_1 = \frac{-0.441 - 1.89 \cdot (-4.9)}{0.3} = 29.4$$

Finally, we find a_0 using the linear equation from row 1. We see

$$1a_0 + 3a_1 + 9a_2 = 3 \qquad \Longrightarrow \qquad a_0 = \frac{3 - 3 \cdot 29.4 - 9 \cdot (-4.9)}{1} = -41.1$$

We used our equivalent system to find a model for our original data given by

$$h(t) = -4.9t^2 + 29.4t - 41.1$$

We can use the method of completing the square to find

$$\begin{aligned} h(t) &= -4.9t^2 + 29.4t - 41.1 \\ &= -4.9(t^2 + 6t + 9 - 9) - 41.1 \\ &= -4.9(t-3)^2 + 3 \end{aligned}$$

This algebraic expression indicates that we probably started our experiment at $t = 3$ seconds dropping our object from an initial height of 3 meters. We also probably used a zero initial velocity. Thus, we would conclude that our model $h(t)$ is valid for the domain $[3, 3.782)$. The upper bound of this domain $t \approx 3.782$s is the point in time when the object hits the ground (assumed to be at zero height). We did not model any bouncing activity, only the drop of the object.

It is important to notice here that the model we constructed EXACTLY matches the data we collected. This is extremely rare and represents an ideal experiment. As we will see, in almost all human experiments, there are many sources of error that introduce noise into the data. Thus, normally when we work to model collected data, we do not solve linear systems problems but instead work with least squares problems. Much more about this subject will be discussed in our work with least squares.

The previous example demonstrates regular Gaussian Elimination as a technique to solve a system of n equations with n unknowns. This specialized technique works well for a very special subclass of squares matrices known as regular matrices.

Definition 5.2: Regular Matrix

Let $A \in \mathbb{R}^{n\times n}$ be a given, square matrix. We say that A is regular if it reduces to an upper-triangular matrix U with all non-zero pivots in the diagonal elements using only multiplication by shear matrices.

As we apply our regular Gaussian Elimination technique to regular matrices, each successive pivot will appear on the diagonal and must be nonzero. In each step, we use the pivot row to introduce zeros in the entries below our current pivot. When the system is fully reduced to upper-triangular form, we then apply Backward Substitution to find the solution to the original system.

It is important to notice that regular matrices are a special subclass of matrices. In general, we will have to use more sophisticated reduction algorithms to deal with some challenges posed by non-regular and non-square matrices. These issues are discussed in the next section.

EXAMPLE 5.1.5

Let's look back at our vector model for Hooke's Law from Example 2.2.6. Recall, that we collected data to track the displacement of the movable end of a spring versus the applied mass. We used vector operations to create a vector model of Hooke's Law given by

$$\mathbf{f} = k\mathbf{e}.$$

Suppose that we generate a linear model for Hooke's law using only two data points:

Mass Number	Measured Mass	Calculated Force	Measured Elongation
1	0.100kg	0.98N	0.051 m
2	0.200kg	1.96N	0.108 m

Then, we can create a linear model

$$f(u) = ke + b$$

for unknown scalars k and b. This experiment yields two linear equations, give by

$$\begin{aligned} 0.051\,k + b &= 0.98 \\ 0.108\,k + b &= 1.96 \end{aligned}$$

In this case, we use our data to create a square linear-systems problem $A\mathbf{x} = \mathbf{b}$ in the following way

$$\begin{bmatrix} 1 & 0.051 \\ 1 & 0.108 \end{bmatrix} \begin{bmatrix} b \\ k \end{bmatrix} = \begin{bmatrix} 0.98 \\ 1.96 \end{bmatrix}$$

We begin our reduction algorithm by identifying our first pivot in entry $(1, 1)$.

STEP 1: Identify the first pivot

Identify the entry with row index 1 and column index 1. Make sure this entry is nonzero and refer to this nonzero entry as the first pivot of our matrix. Refer to column 1 as the first pivot column.

In this case we see $a_{11} = 1$ is nonzero. Thus, we do not need to interchange rows. Instead, we call the first column of this matrix our first pivot columns and proceed with our elimination process.

$$\begin{bmatrix} 1 & 0.051 \\ 1 & 0.108 \end{bmatrix}$$

STEP 2: Create zeros in all entries below the first pivot

Multiply the original system by a sequence of shear matrices to introduce zeros in all entries below the first pivot.

In this case, we want to introduce a zero to entry a_{21}.

$$\begin{bmatrix} 1 & 0.051 \\ 1 & 0.108 \end{bmatrix}$$

We see that if we add -1 time row 1 to 1 times row 2 and put the result into row 2, we will get a zero in entry $(2,1)$:

$$-1 \cdot \begin{bmatrix} 1 & 0.051 \end{bmatrix} + \begin{bmatrix} 1 & 0.108 \end{bmatrix} = \begin{bmatrix} 0 & 0.057 \end{bmatrix}$$

We can accomplish this same transformation using multiplication with shear matrix $S_{21}(-1)$ as follows

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.051 \\ 1 & 0.108 \end{bmatrix} \begin{bmatrix} b \\ k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.98 \\ 1.96 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0.051 \\ 0 & 0.057 \end{bmatrix} \begin{bmatrix} b \\ k \end{bmatrix} = \begin{bmatrix} 0.98 \\ 0.98 \end{bmatrix}$$

STEP 3: Identify next pivot and introduce zeros under pivot

Move to the next row down and next column to the right. Call the first nonzero entry in this row the second pivot. The column that this pivot is in is called the second pivot column.

Because we started with a 2×2 coefficient matrix A, we have transformed our system into the equivalent upper-triangular system and we can now solve this system using backward substitution.

In addition to solving problems involving upper-triangular matrices, we also will use lower-triangular coefficient matrices. In this case, we have the square linear-systems problem

$$L \cdot \mathbf{y} = \mathbf{b}$$

EXAMPLE 5.1.6

Let's look at a system of 4 linear equations in 4 unknowns with a lower-triangular coefficient matrix L. Suppose that the diagonal elements of L are nonzero and consider the linear system

$$\begin{bmatrix} \ell_{11} & 0 & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & \ell_{44} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Now, we can focus on the scalar versions of this matrix equation by looking at the individual row entries of the left- and right-hand side. We see:

$$\begin{aligned} \ell_{11}y_1 + \quad 0 + \quad 0 + \quad 0 &= b_1 \\ \ell_{21}y_1 + \ell_{22}y_2 + \quad 0 + \quad 0 &= b_2 \\ \ell_{31}y_1 + \ell_{32}y_2 + \ell_{33}y_3 + \quad 0 &= b_3 \\ \ell_{41}y_1 + \ell_{42}y_2 + \ell_{43}y_3 + \ell_{44}y_4 &= b_4 \end{aligned}$$

We see that the last equation has only one unknown. Moreover, if $u_{ii} \neq 0$ for all values of i, we solve for the unknown

$$y_1 = \frac{1}{\ell_{55}} \cdot (b_1)$$

We now know the value of coefficient y_1 and we have eliminated one of our unknowns. We move up to the second equation

$$\ell_{21}y_1 + \ell_{22}y_2 + = b_2, \qquad \Longrightarrow \qquad y_2 = \frac{1}{\ell_{22}} \cdot (b_4 - \ell_{21}y_1)$$

We continue with row 3 and solve for y_3 by recognizing

$$\ell_{31}y_1 + \ell_{32}y_2 + \ell_{33}y_3 = b_3 \qquad \Longrightarrow \qquad y_3 = \frac{1}{\ell_{33}} \cdot (b_3 - \ell_{31}y_1 - \ell_{32}y_2)$$

$$\Longrightarrow \qquad y_3 = \frac{1}{\ell_{33}} \cdot \left(b_3 - \sum_{j=1}^{2} \ell_{3j}y_j \right)$$

Finally, we find y_4 using the formulas

$$\ell_{41}y_1 + \ell_{42}y_2 + \ell_{43}y_3 + \ell_{44}y_4 = b_4 \quad \Longrightarrow \quad y_4 = \frac{1}{\ell_{44}} \cdot (b_4 - \ell_{41}y_1 - \ell_{42}y_2 - \ell_{43}y_3)$$

$$\Longrightarrow \quad y_2 = \frac{1}{\ell_{44}} \cdot \left(b_4 - \sum_{j=1}^{3} u_{4j}y_j \right)$$

We have now solved our entire linear systems problem for unknowns y_i. This process, described in general below, will be known as forward substitution.

Theorem 21: Forward Substitution: Lower-Triangular $L\mathbf{x} = \mathbf{b}$

Let $L\mathbf{x} = \mathbf{b}$ be a given linear-systems problem with lower-triangular $L \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$. If $u_{ii} \neq 0$ for all $i \in \{1, 2, ..., n\}$, then our linear-system problem has a unique solution. Further, we can find this solution using the following algorithm:

$$x_1 = \frac{b_1}{u_{11}}$$

$$x_i = \frac{1}{u_{ii}}\left(b_i - \sum_{j=1}^{i-1} u_{ij}\, x_j\right)$$

where $i = 2, 3, ..., n$.

One of the major features of linear algebra is that systems of linear equations are equivalent to matrix equations. That is, we can use algebra on matrices as a method to solve systems of linear equations. Similarly, any matrix equation can be realized as a linear systems equation.

Definition 5.3: Coefficient Matrix of a linear system

Given a linear system of m equations and n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

we can write this system as a linear systems problem $A\mathbf{x} = \mathbf{b}$ where the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is known as the **coefficient matrix** of our linear system.

The row dimension of the matrix A indicates the number of simultaneous equations to be solve while the column dimension of A counts the number of unknown variables in our linear system.

The basic idea of solving a linear system of equations is to transform the linear system into an equivalent matrix equation, then perform invertible matrix computations on the matrix equation to replace the given linear system with an equivalent linear system that is much easier to solve.

Definition 5.4: Fundamental Questions about Linear Systems

- **Existence Question:** Is the system of equations consistent: does the linear system have at least one solution?
- **Uniqueness Question:** If a solution exists, is it the only one: does the linear system have a unique solution?

Definition 5.5: Solution Set to Square Linear-Systems Problem

Suppose we are given a linear-systems problem

$$A\mathbf{x} = \mathbf{b}$$

with $A \in R^{n \times n}$ and $\mathbf{b} \in \mathbb{R}^n$. Then the *solution set* to this linear system is the set of all possible solutions $\mathbf{x} \in \mathbb{R}^n$ that solves our given linear system. In other words, the solution set is given by

$$\{\mathbf{x} \in \mathbb{R}^n \, : \, A\mathbf{x} = \mathbf{b}\}$$

Theorem 22

Let matrix $A \in \mathbb{R}^{n \times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$ be given. Suppose we want to find the solution set to the linear-systems problem $A\mathbf{x} = \mathbf{b}$ Then, there are only three possible scenarios:

i. An empty solution set: no exact solutions exist

ii. A solution set containing one element: a unique solution exists

iii. A solution set with infinite solutions: non-unique solutions exist

Moreover, if A is nonsingular, the square linear-systems problem will always have a unique solution.

Lesson 12: Nonsingular Linear-Systems Problems- Suggested Problems

1. Let $A \in \mathbb{R}^{5\times 5}$ and $\mathbf{b} \in \mathbb{R}^5$ be given by
$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ 12 \\ 54 \\ 6 \end{bmatrix}.$$
   A. Find $\mathbf{x} \in \mathbb{R}^5$ such that $A\mathbf{x} = \mathbf{b}$.
   B. Is this solution unique? How do you know?

2. Let $A \in \mathbb{R}^{4\times 4}$ and $\mathbf{b} \in \mathbb{R}^4$ be given by
$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}.$$
   A. Find $\mathbf{x} \in \mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{b}$ using backward substitution. Show all your steps, one-by-one.
   B. Is this solution unique? How do you know?

3. Let $A \in \mathbb{R}^{3\times 3}$ and $\mathbf{b} \in \mathbb{R}^3$ be given by
$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 5 & -4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}.$$
   A. Find $\mathbf{x} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{b}$ using forward substitution. Show all your steps, one-by-one.
   B. Is this solution unique? How do you know?

4. Re-solve the problem given in-class:
$$\begin{bmatrix} 1 & 3 & 9.00 \\ 1 & 3.3 & 10.89 \\ 1 & 3.6 & 12.96 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3.000 \\ 2.559 \\ 1.234 \end{bmatrix}$$

5. Exercise 1.1.1 pg 10
6. Exercise 1.1.3 pg 10
7. Exercise 1.1.11 pg 10

8. Exercise 1.1.13 pg 10
9. Exercise 1.1.15 p. 10
10. Exercise 1.1.28 pg 11
11. Exercise 1.1.29 pg 11
12. Exercise 1.1.30 pg 11
13. Exercise 1.1.31 p. 11
14. Exercise 1.1.32 p. 11
15. **Building a Better Roller Coaster:** From Stewart's Calculus

    Remark: (This problem is designed students who want to earn above a 90%.)

    The problem below demonstrates a simplifies "real-world" application that gives rise to an 11×11 linear system of equations. This is an example of a much more general field of cubic spline interpolation. For interested readers, I highly value the following text on this subject: A Practical Guide to Splines by Carl de Boor.

    Suppose you are asked to design the first ascent and drop for a new roller coaster. By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop -1.6. You decide to connect these two straight stretches $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola $y = f(x) = ax^2 + bx + c$, where x and $f(x)$ are measured in feet. For the track to be smooth there can't be abrupt changes in direction, so you want the linear segments L_1 and L_2 to be the tangent to the parabola at the transition points P and Q. To simplify the equations, you decide to place the origin at point P.

    A. Suppose the horizontal distance between P and Q is 100 ft. Write equations in a, b and c that will ensure that the track is smooth at the transition points.
    B. Solve the equations in part (a) for a, b and c to find a formula for $f(x)$.
    C. Plot $L_1(x), f,$ and $L_2(x)$ to verify graphically that the transitions are smooth.
    D. Find the difference in elevation between P and Q.
    E. The solution to problem 5 might *look* smooth, but it might not *feel* smooth because the piecewise defined function [consisting of $L_1(x)$ for $x < 0$, $f(x)$ for $0 \leq x \leq 100$, and $L_2(x)$ for $x > 100$] doesn't have a continuous second derivative. You decide to improve the design by using a quadratic function $q(x) = ax^2 + bx + c$ only on the interval $10 \leq x \leq 90$ and connecting it to the linear functions by means of two cubic functions:

    $$\begin{aligned} g(x) &= kx^3 + lx^2 + mx + n, & 0 \leq x < 10 \\ h(x) &= px^3 + qx^2 + rx + s, & 90 < x <\leq 100 \end{aligned}$$

    F. Write a system of equations in 11 unknowns that ensure that the functions and their first two derivatives agree at the transition points.

