Answers **Lesson 11 Warm Up Quiz**

Math 2B: Applied Linear Algebra

True/False

For the problems below, circle T if the answer is true and circle F is the answer is false.

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1. T **F** Any two matrices that are conformable for matrix multiplication must have the same number of rows.

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2. T **F** Let $A \in \mathbb{R}^{9\times 6}$ and $X \in \mathbb{R}^{6\times 7}$. Set

$$B = A \cdot X.$$

Then $B(:,3) = A(3,:) \cdot X$

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3. T **F** Every square matrix is a product of elementary matrices.

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4. T **F** If $A \in \mathbb{R}^{n\times n}$ is a diagonal matrix with nonzero diagonal entries, and $B \in \mathbb{R}^{m\times n}$, then multiplying B on the right by A scales the rows of B.

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5. **T** F If A and B are $m \times n$ matrices, then the matrix products AB^T and $A^T B$ are defined.

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6. T **F** For rectangular matrices A, B, C, with proper dimensions, if $AB = C$ and C has 2 columns, then A has two columns.

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7. T **F** If $A, B \in \mathbb{R}^{n\times n}$, then $(A - B)(A + B) = A^2 - B^2$.

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8. T **F** For matrices B, C, D with proper dimensions, if $BC = BD$, then $C = D$.

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9. T **F** For any $A, B \in \mathbb{R}^{n\times n}$, $AB = BA$.

Multiple Choice

For the problems below, circle the correct response for each question.

1. Define the matrix $B \in \mathbb{R}^{4\times 4}$ by the following product:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Let $\mathbf{e}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$ and $\mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$. Using this definition, we see that $b_{23} = \mathbf{e}_2^T B \mathbf{e}_3$ is given by which of the following:

A. $b_{23} = a_{43}$ B. $b_{23} = a_{24}$ C. $b_{23} = a_{23}$ D. $b_{23} = a_{44}$ E. None of these.

2. Let $A \in \mathbb{R}^{3\times 3}$ be defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where A is nonsingular and $a_{11} \neq 0$. Suppose that we choose $E \in \mathbb{R}^{3\times 3}$ such that the product EA has the following structure:

$$EA = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

In this case, the symbol $*$ represents a real numbers. Then, E must be given by which of the following:

A. $E = S_{13}\left(-\frac{a_{13}}{a_{11}}\right) \cdot S_{13}\left(-\frac{a_{12}}{a_{11}}\right)$

B. $E = S_{31}\left(\frac{a_{31}}{a_{11}}\right) \cdot S_{21}\left(\frac{a_{12}}{a_{11}}\right)$

C. $E = S_{31}\left(-\frac{a_{31}}{a_{11}}\right) \cdot S_{21}\left(-\frac{a_{12}}{a_{11}}\right)$

D. $E = S_{31}\left(-\frac{a_{11}}{a_{31}}\right) \cdot S_{21}\left(-\frac{a_{11}}{a_{21}}\right)$

E. None of these

3. Define the matrix $B \in \mathbb{R}^{4\times4}$ by the following product:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Using this definition, we see that b_{42} is given by which of the following:

A. $b_{42} = a_{32}$ B. $b_{42} = a_{42}$ C. $b_{42} = a_{24}$ **D.** $b_{42} = a_{34}$ E. $b_{42} = a_{43}$

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4. Let $A \in \mathbb{R}^{12\times7}$ and $B \in \mathbb{R}^{12\times6}$. Suppose $C = B^T A$. What are the dimensions of $C(:,2)$?

A. 7×1 **B.** 6×1 C. 6×7 D. 7×6 E. 1×6

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5. Which of the following represents the matrix-matrix product: $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3.3 & 10.89 \\ 1 & 3.6 & 12.96 \end{bmatrix}$:

A. $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 3.3 & 1.89 \\ 0 & 3.6 & 3.96 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 3.3 & 10.89 \\ 0 & 3.6 & 12.96 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 3.3 & 1.89 \\ 0 & 0 & 3.96 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 0.3 & 10.89 \\ 0 & 0.6 & 12.96 \end{bmatrix}$ **E.** $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 0.3 & 1.89 \\ 0 & 0.6 & 3.96 \end{bmatrix}$

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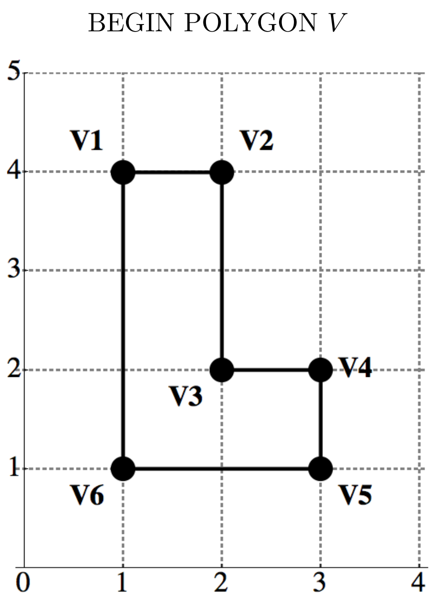
6. Let matrix $P \in \mathbb{R}^{4\times5}$ be given as follows:

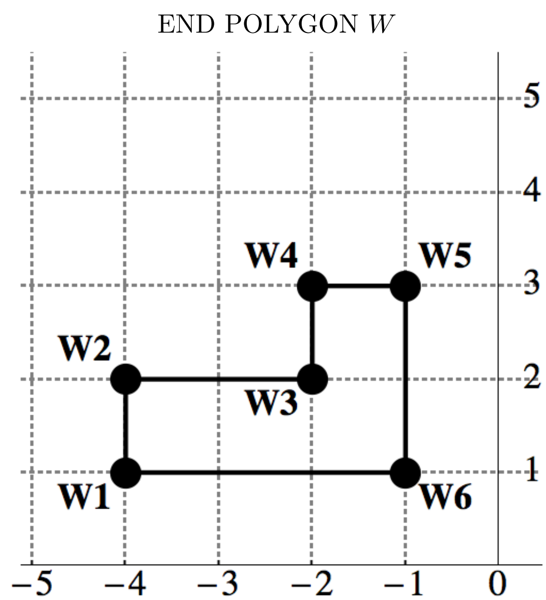
$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using this definition, we see that a_{12} is equal to which of the following:

A. $a_{12} = p_{21}$ B. $a_{12} = p_{24}$ C. $a_{12} = p_{42}$ D. $a_{12} = p_{12}$ **E.** $a_{12} = p_{25}$

For the two problems below, consider the polygons V and W.:





7. Which of the following vertex matrices V encodes the begin polygon above? For this model, assume that the kth column of V encodes vertex Vk, for $k \in \{1, 2, 3, 4, 5, 6\}$:

A. $\begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 4 & 4 & 2 & 2 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 & 2 & 3 & 3 & 1 \\ 4 & 4 & 2 & 2 & 1 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 4 & 4 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 3 & 3 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 4 & 4 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 3 & 3 \end{bmatrix}$

E. $\begin{bmatrix} -4 & -4 & -2 & -2 & -1 & -1 \\ 1 & 2 & 2 & 3 & 3 & 1 \end{bmatrix}$

8. As noted above, let V be the vertex matrix that models the begin polygon and W be the vertex matrix that models the end polygon. Which matrix Q below satisfies equation

$$W = QV$$

A. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

B. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

9. Define matrix A by

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 & 6 \\ 1 & -2 & 3 & 2 & 0 \\ -4 & 1 & 0 & 5 & 7 \\ 6 & -2 & 8 & 0 & -1 \\ -7 & -2 & -1 & 3 & 1 \end{bmatrix}$$

For which of the following matrices E below will the matrix product

$$EA = C$$

not have a zero in the first column?

A. $S_{21}(-0.5)$ B. $S_{31}(2)$ C. $S_{41}(-3)$ D. $S_{51}(3.5)$ **E.** $S_{41}(3)$

10. Let $A \in \mathbb{R}^{8\times 4}$, $B \in \mathbb{R}^{4\times 7}$, and $C \in \mathbb{R}^{7\times 5}$. Let the matrix D be formed by the product

$$D = (A \cdot B \cdot C)^T$$

What are the dimensions of the matrix $[D(:,4)]^T$?

A. 8×5 B. 5×8 **C.** 1×5 D. 1×8 E. 5×1

For the next two problems below, consider the wireframe model for a begin polygon V defined by vertex matrix and edge table below.

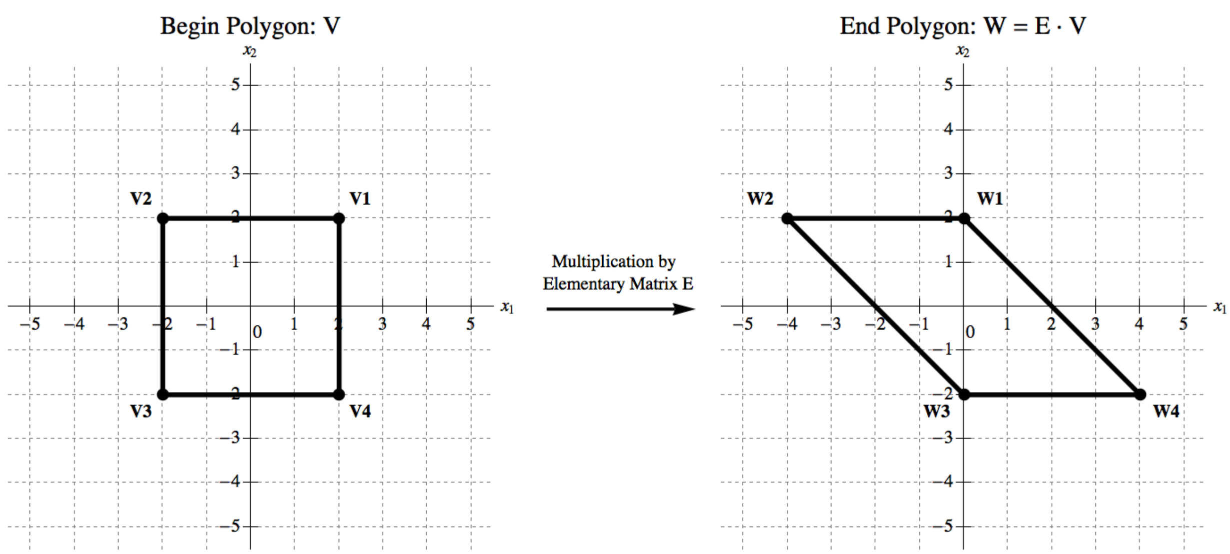
$$V = \begin{bmatrix} 2 & -2 & -2 & 2 \\ 2 & 2 & -2 & -2 \end{bmatrix}$$

Edge #	Start Vertex	End Vertex
1	1	2
2	2	3
3	3	4
4	4	1

Let W be a wireframe model for an end polygon given by

$$W = \begin{bmatrix} 0 & -4 & 0 & 4 \\ 2 & 2 & -2 & -2 \end{bmatrix}$$

Assume W formed by multiplying V by some matrix $E \in \mathbb{R}^{2\times 2}$ with $W = E \cdot V$. Also, assume that the edge tables of V and W are identical. Under these assumptions, the wireframe model for both V and W are given below.



11. Choose the matrix E used to produce W in this situation:

A. $S_{21}(-2)$ B. $S_{12}(-2)$ C. $S_{21}(-1)$ D. $S_{12}(1)$ **E.** $S_{12}(-1)$

12. Find the length of edge 4 from the wireframe model for the end polygon $W = E \cdot V$ in the problem above.

A. 2 B. $\sqrt{20}$ C. 4 **D.** $4\sqrt{2}$ E. 0

13. Let matrix $B \in \mathbb{R}^{4\times 4}$ be given as follows:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

In symbols, we can write

$$B = S_{41}(-2) \cdot A \cdot D_4(3)$$

Using this definition, we see that b_{44} is equal to which of the following:

A. $-6a_{44}$ B. $-2a_{14} + 3a_{44}$ **C.** $-6a_{14} + 3a_{44}$ D. $3a_{14} - 6a_{44}$ E. $3a_{14} - 2a_{44}$

14. Let $n \in \mathbb{N}$ with $n \geq 3$. Suppose that we define the matrix

$$B = I_n + c_1\,\mathbf{e}_2\,\mathbf{e}_1^T - c_2\,\mathbf{e}_3\,\mathbf{e}_1^T$$

where $\mathbf{e}_k = I_n(:,k)$. Which of the following is equivalent to B^{-1}?

A. $S_{21}(c_1) \cdot S_{31}(-c_2)$ B. $S_{31}(c_2) - S_{21}(c_1)$ **C.** $S_{31}(c_2) \cdot S_{21}(-c_1)$

D. $S_{21}\left(\dfrac{1}{c_1}\right) \cdot S_{31}\left(\dfrac{-1}{c_2}\right)$ E. $S_{12}(c_1) \cdot S_{13}(-c_2)$

Free Response

1. Write the definition for the column-partition version of matrix-matrix multiplication.

Solution: Let $A \in \mathbb{R}^{m\times n}$ and $X \in \mathbb{R}^{n\times p}$. The matrix-matrix product

$$B = A \cdot X$$

is the matrix $B \in \mathbb{R}^{m\times p}$. The matrix-matrix multiplication operation used calculate this product is a map between vector spaces $\cdot : \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times p} \longrightarrow \mathbb{R}^{m\times p}$ with two inputs.

The **left argument** of the matrix product

$$A \cdot X$$

is the matrix $A \in \mathbb{R}^{m\times n}$ on the left side of the multiplication sign. On the other hand, the **right argument** of this product is matrix $X \in \mathbb{R}^{n\times p}$ on the right side of the multiplication sign.

We say that we multiply A on the right by X if A is the left argument and X is the right argument. To execute multiplication of A on the right by X, the number of columns of the left matrix A must be equal to the number of rows of the right matrix X. If the column dimension of A equals the row dimension of X, we say that A is **conformable for right multiplication** by X. Another way to say this is that the inner dimensions of the product agree. If the dimensions of A suitable to multiply on the right by X, we say that matrix A is **nonconformable** for matrix multiplication on the right by X.

Definition 1: Matrix-matrix multiplication by columns

Let $A \in \mathbb{R}^{m \times n}$ and $X \in \mathbb{R}^{n \times p}$. If we multiply A on the right by X to form the $m \times p$ matrix $B = A \cdot X$, then

$$\text{Column}_k(B) = A \cdot \text{Column}_k(X),$$

for $k \in \{1, 2, ..., p\}$. In other words, the kth column of B is the matrix A multiplied on the right by the kth column of X. This operation is written using colon notation as

$$B(:,k) = A \cdot X(:,k) = \sum_{j=1}^{n} x_{jk}\, A(:,j).$$

The kth column of the product $B = A \cdot X$ is a linear combination of the columns of matrix A with scalar weights defined by the individual entries in the kth column of X. We can write the kth column of the product B is an $m \times 1$ column with

$$B(:,k) = \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{bmatrix} = x_{1k} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_{2k} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_{nk} \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Using this definition, we execute matrix-matrix multiplication one column at a time to build the output matrix B.

2. Let $A \in \mathbb{R}^{4\times 4}$. Multiply A on the right by a matrix X to achieve each of the operations below. In each case, specifically state the entry-by-entry definition of the matrix X used to accomplish these operations.

   A. Double column 1

Solution: Define a 4×4 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Suppose we wish to double column one of A and leave the other columns untouched. To this end, we multiply A on the right-hand side by an appropriately sized dilation matrix:

$$B = A \cdot D_1(2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let's begin by calculating the first column of the product

$$B(:,1) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} 2\,a_{11} \\ 2\,a_{21} \\ 2\,a_{31} \\ 2\,a_{41} \end{bmatrix}$$

Next, calculate $\text{Column}_2(B) = A \cdot \text{Column}_2\Big(D_1(2)\Big)$ which is a linear combination of the columns of A with scaling coefficients coming from the second column of $D_1(2)$.

$$B(:,2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 1 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix}$$

We continue by calculating $\text{Column}_3(B) = A \cdot \text{Column}_3\Big(D_1(2)\Big)$ written as

$$B(:,3) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 1 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix}$$

Finally, we calculate the fourth column of B

$$B(:,4) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 1 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix}$$

We have now constructed a column partition of B. We can create the entire matrix B by combining together our three matrix-vector products

$$B = \begin{bmatrix} 2\,a_{11} & a_{12} & a_{13} & a_{14} \\ 2\,a_{21} & a_{22} & a_{23} & a_{24} \\ 2\,a_{31} & a_{32} & a_{33} & a_{34} \\ 2\,a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

We see that right multiplication by $D_1(2)$ scales the first column of A and leaves all other columns untouched.

B. Interchange columns 1 and 4

Solution: Let's use matrix-matrix multiplication to permute the columns 1 and 4 of a matrix. To this end, let $A \in \mathbb{R}^{4\times 4}$. Let P_{14} be the transposition matrix generated by swapping the first and fourth column of the identity matrix. Then consider

$$B = A \cdot P_{14} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In this case, let's use the column-partition version of matrix-matrix multiplication to find each column of B. To this end consider, calculate $\text{Column}_1(B) = A \cdot \text{Column}_1\left(P_{14}\right)$ as

$$B(:,1) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 1 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix}.$$

Now, the second column of B is $\text{Column}_2(B) = A \cdot \text{Column}_2\left(P_{14}\right)$ given by

$$B(:,2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 1 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix}.$$

The third column $\text{Column}_3(B) = A \cdot \text{Column}_3\left(P_{14}\right)$ of our product is

$$B(:,3) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 1 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix}.$$

Finally, the last column $\text{Column}_4(B) = A{\cdot}\text{Column}_4\left(P_{14}\right)$ is a linear combination of the columns of A with scalars defined by the entries in the fourth column of P_{14}. This column vector is calculated as follows

$$B(:,4) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}.$$

Matrix B results from combining each of these outputs and is given by

$$B = \begin{bmatrix} a_{14} & a_{12} & a_{13} & a_{11} \\ a_{24} & a_{22} & a_{23} & a_{21} \\ a_{34} & a_{32} & a_{33} & a_{31} \\ a_{44} & a_{42} & a_{43} & a_{41} \end{bmatrix}$$

C. Add 2 times column 2 to column 3

Solution: Let's use matrix-matrix multiplication to add 2 times column 2 to column 3. To this end, let $A \in \mathbb{R}^{4\times 4}$. Let $S_{23}(2)$ be a 4×4 shear matrix and consider

$$B = A \cdot S_{23}(2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case, let's use the column-partition version of matrix-matrix multiplication to find each column of B. To this end consider, calculate $\text{Column}_1(B) = A \cdot \text{Column}_1\Big(P_{14}\Big)$ as

$$B(:,1) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}.$$

Now, the second column of B is $\text{Column}_2(B) = A \cdot \text{Column}_2\Big(P_{14}\Big)$ given by

$$B(:,2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 1 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix}.$$

The third column $\text{Column}_3(B) = A \cdot \text{Column}_3\Big(P_{14}\Big)$ of our product is

$$B(:,3) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 2 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 1 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} 2\,a_{12} + a_{13} \\ 2\,a_{22} + a_{23} \\ 2\,a_{32} + a_{33} \\ 2\,a_{42} + a_{43} \end{bmatrix}.$$

Finally, the last column $\text{Column}_4(B) = A \cdot \text{Column}_4\Big(P_{14}\Big)$ is a linear combination of the columns of A with scalars defined by the entries in the fourth column of P_{14}. This column vector is calculated as follows

$$B(:,4) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 1 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix}.$$

Matrix B results from combining each of these outputs and is given by

$$B = \begin{bmatrix} a_{11} & a_{12} & 2\,a_{12} + a_{13} & a_{14} \\ a_{21} & a_{22} & 2\,a_{22} + a_{23} & a_{24} \\ a_{31} & a_{32} & 2\,a_{32} + a_{33} & a_{34} \\ a_{41} & a_{42} & 2\,a_{42} + a_{43} & a_{44} \end{bmatrix}$$

D. Delete column 4 (so that the column dimension is reduced by 1)

Solution: Let's use matrix-matrix multiplication to delete column 4 of matrix A. We will call this operation **deflation**, since the product is a "deflated" version of the original matrix A that is one column smaller. To this end, let $A \in \mathbb{R}^{4\times 4}$. Let $D_4 \in \mathbb{R}^{4\times 3}$ be matrix that results by deleting the fourth column of I_4. Then, multiply A on the right by D_4 to form the product

$$B = A \cdot D_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

In this case, let's use the column-partition version of matrix-matrix multiplication to find each column of B. To this end consider, calculate $\text{Column}_1(B) = A \cdot \text{Column}_1\Big(P_{14}\Big)$ as

$$B(:,1) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}.$$

Now, the second column of B is $\text{Column}_2(B) = A \cdot \text{Column}_2\Big(P_{14}\Big)$ given by

$$B(:,2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 1 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix}.$$

The third and final column $\text{Column}_3(B) = A \cdot \text{Column}_3\Big(P_{14}\Big)$ of our product is

$$B(:,3) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} + 1 \cdot \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} + 0 \cdot \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \end{bmatrix} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix}.$$

Matrix B results from combining each of these outputs and is given by

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

3. Write the definition for the row-partition version of matrix-matrix multiplication.

Solution:

The **left argument** of the matrix product

$$A \cdot X$$

is the matrix $A \in \mathbb{R}^{m\times n}$ on the left side of the multiplication sign. On the other hand, the **right argument** of this product is matrix $X \in \mathbb{R}^{n\times p}$ on the right side of the multiplication sign.

Sometimes we want to construct a matrix-matrix product by rows instead of by columns. We say that we multiply X on the left by A if X is the right argument and A is the left argument. To execute multiplication of X on the left by A, the number of rows of the right matrix X must equal the number of columns of the left matrix A. If the row dimension of X equals the column dimension of A, we say that X is **conformable for left multiplication** by A. Again, we can say that the inner dimensions of the product agree. If the dimensions of X are not suitable to multiply on the right by A, we say that matrix X is **nonconformable for multiplication on the right** by X.

Definition 2: Matrix-matrix multiplication by rows

Let $A \in \mathbb{R}^{m\times n}$ and $X \in \mathbb{R}^{n\times p}$. If we multiply X on the left by A to form the $m \times p$ matrix $B = A \cdot X$, then

$$\text{Row}_i(B) = \text{Row}_i(A) \cdot X,$$

for $i \in \{1, 2, ..., m\}$. In other words, the ith row of B is the matrix X multiplied on the left by the ith row of A. This operation is written using colon notation as

$$B(i, :) = A(i, :) \cdot X = \sum_{j=1}^{m} a_{ij} X(j, :)$$

The ith row of the product $B = A \cdot X$ is a linear combination of the rows of matrix X with scalar weights defined by the individual entries in the ith row of A. Using this definition, we execute matrix-matrix multiplication one row at a time to build the individual rows of the output matrix B. For the ith row of the product, we calculate

$$\begin{aligned}\begin{bmatrix} b_{i1} & b_{i2} & \cdots & b_{ip} \end{bmatrix} &= a_{i1} \cdot \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \end{bmatrix} \\ &+ a_{i2} \cdot \begin{bmatrix} x_{21} & x_{22} & \cdots & x_{2p} \end{bmatrix} \\ &\vdots \\ &+ a_{in} \cdot \begin{bmatrix} x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix}\end{aligned}$$

In such a way we construct the row partitions version of B.

4. Let $X \in \mathbb{R}^{4\times 4}$. Multiply X on the left by a matrix A to achieve each of the operations below. In each case, specifically state the entry-by-entry definition of the matrix A used to accomplish these operations.

   A. Halve row 3

Solution: Define a 4×4 matrix

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Suppose we wish to halve row three of X and leave the other columns untouched. To this end, we multiply X on the left-hand side by an appropriately sized dilation matrix:

$$B = D_3\left(\frac{1}{2}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Let's begin by calculating the first row $\text{Row}_1(B) = \text{Row}_1\left(D_3\left(\frac{1}{2}\right)\right) \cdot X$ given by

$$B(1,:) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= 1 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

Solution: Next, calculate $\text{Row}_2(B) = \text{Row}_2\left(D_3\left(\frac{1}{2}\right)\right) \cdot X$ which is a linear combination of the rows of X with scaling coefficients coming from the second row of $D_3\left(\frac{1}{2}\right)$.

$$B(2,:) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= 0 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 1 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

We continue by calculating $\text{Row}_3\left(D_3\left(\frac{1}{2}\right)\right) \cdot X$ written as

$$B(3,:) = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= 0 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ \frac{1}{2} \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\,x_{31} & \frac{1}{2}\,x_{32} & \frac{1}{2}\,x_{33} & \frac{1}{2}\,x_{34} \end{bmatrix}$$

Finally, we calculate the fourth column of B

$$B(4,:) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= 0 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 1 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

We have now constructed a row partition of B. We can create the entire matrix B by combining together our four rows together as

$$B = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ \frac{1}{2}\, x_{31} & \frac{1}{2}\, x_{32} & \frac{1}{2}\, x_{33} & \frac{1}{2}\, x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

We see that left multiplication by $D_3\left(\frac{1}{2}\right)$ scales the third row of X by a factor of 0.5 and leaves all other rows untouched.

B. Add row 2 to row 4

Solution: Suppose we wish to add 1 times row 2 of X to row four and leave the other columns untouched. To this end, we multiply X on the left-hand side by an appropriately-sized shear matrix:

$$B = S_{42}(1) \cdot X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Let's begin by calculating the first row $\text{Row}_1(B) = \text{Row}_1(S_{42}(1)) \cdot X$ given by

$$\begin{aligned} B(1,:) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \\ &= 1 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix} \\ &+ 0 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \\ &+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \\ &+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \\ &= \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix} \end{aligned}$$

Next, calculate $\text{Row}_2(B) = \text{Row}_2(S_{42}(1)) \cdot X$ which is a linear combination of the rows of X with scaling coefficients coming from the second row of $D_3\left(\frac{1}{2}\right)$.

$$\begin{aligned} B(2,:) &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \\ &= 0 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix} \\ &+ 1 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \\ &+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \\ &+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \\ &= \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \end{aligned}$$

We continue by calculating $\text{Row}_3\left(S_{42}(1)\right) \cdot X$ written as

$$B(3,:) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= 0 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ 1 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

Finally, we calculate the last row $\text{Row}_4\left(S_{42}(1)\right) \cdot X$ written as

$$B(4,:) = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= 0 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 1 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 1 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} 1\,x_{21} + x_{41} & 1\,x_{22} + x_{42} & 1\,x_{23} + x_{43} & 1\,x_{24} + x_{44} \end{bmatrix}$$

We have now constructed a row partition of B. We can create the entire matrix B by combining together our four rows together as

$$B = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ 1\,x_{21} + x_{41} & 1\,x_{22} + x_{42} & 1\,x_{23} + x_{43} & 1\,x_{24} + x_{44} \end{bmatrix}$$

We see that left multiplication by $S_{42}\left(1\right)$ adds the second row column of X to the fourth row of X and leaves all other rows untouched.

C. Swap rows 1 and 2

Solution: Suppose we wish to switch rows 1 and 2 of X. To this end, we multiply X on the left-hand side by an appropriately-sized permutation matrix:

$$B = P_{12} \cdot X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Let's begin by calculating the first row $\mathrm{Row}_1(B) = \mathrm{Row}_1\left(P_{12}\right) \cdot X$ given by

$$\begin{aligned} B(1,:) &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \\ &= 0 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix} \\ &+ 1 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \\ &+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \\ &+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \\ &= \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \end{aligned}$$

Next, calculate $\mathrm{Row}_2(B) = \mathrm{Row}_2\left(P_{12}\right) \cdot X$ which is a linear combination of the rows of X with scaling coefficients coming from the second row of $D_3\left(\frac{1}{2}\right)$.

$$\begin{aligned} B(2,:) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \\ &= 1 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix} \\ &+ 0 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \\ &+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix} \\ &+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \\ &= \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix} \end{aligned}$$

We continue by calculating $\mathrm{Row}_3(P_{12}) \cdot X$ written as

$$B(3,:) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= 0 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ 1 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

Finally, we calculate the last row $\mathrm{Row}_4(P_{12}) \cdot X$ written as

$$B(4,:) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= 0 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 1 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

We have now constructed a row partition of B. We can create the entire matrix B by combining together our four rows together as

$$B = \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

We see that left multiplication by P_{12} swaps rows 1 and 2 of the matrix X and leaves all other rows untouched.

D. Subtract row 1 from each of the other rows

Solution: Suppose we wish to subtract rows 1 form all other rows. To this end, we multiply X on the left-hand side by an appropriately-sized unit lower triangular matrix:

$$B = L_1 \cdot X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

Let's begin by calculating the first row $\text{Row}_1(B) = \text{Row}_1(L_1) \cdot X$ given by

$$B(1,:) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= 1 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

Next, calculate $\text{Row}_2(B) = \text{Row}_2(P_{12}) \cdot X$ which is a linear combination of the rows of X with scaling coefficients coming from the second row of $D_3\left(\frac{1}{2}\right)$.

$$B(2,:) = \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= -1 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 1 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} -1\,x_{11} + x_{21} & -1\,x_{12} + x_{22} & -1\,x_{13} + x_{23} & -1\,x_{14} + x_{24} \end{bmatrix}$$

We continue by calculating $\text{Row}_3(L_1) \cdot X$ written as

$$B(3,:) = \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= -1 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ 1 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} -1\,x_{11} + x_{31} & -1\,x_{12} + x_{32} & -1\,x_{13} + x_{33} & -1\,x_{14} + x_{34} \end{bmatrix}$$

Finally, we calculate the last row $\text{Row}_4(L_1) \cdot X$ written as

$$B(3,:) = \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= -1 \cdot \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

$$+ 0 \cdot \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}$$

$$+ 1 \cdot \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix}$$

$$= \begin{bmatrix} -1\,x_{11} + x_{41} & -1\,x_{12} + x_{42} & -1\,x_{13} + x_{43} & -1\,x_{14} + x_{44} \end{bmatrix}$$

We have now constructed a row partition of B. We can create the entire matrix B by combining together our four rows together as

$$B = \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \\ -1\,x_{11} + x_{21} & -1\,x_{12} + x_{22} & -1\,x_{13} + x_{23} & -1\,x_{14} + x_{24} \\ -1\,x_{11} + x_{31} & -1\,x_{12} + x_{32} & -1\,x_{13} + x_{33} & -1\,x_{14} + x_{34} \\ -1\,x_{11} + x_{41} & -1\,x_{12} + x_{42} & -1\,x_{13} + x_{43} & -1\,x_{14} + x_{44} \end{bmatrix}$$

We see that left multiplication by L_1 subtracts row 1 from all other rows of X.