Name : ________________ **Lesson 10 Warm Up Quiz** **Class Number:** _____

Math 2B: Applied Linear Algebra

True/False For the problems below, circle T if the answer is true and circle F is the answer is false.

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1. T F Let $A \in \mathbb{R}^{m \times n}$. If $f(\mathbf{x}) = A\mathbf{x}$, then

$$\text{Rng}(f) = \{\mathbf{b} \in \mathbb{R}^m \,:\, \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

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2. T F When multiplying matrix A and vector $\mathbf{x}$, it is important to check that the inner dimensions agree.

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3. T F If $f(\mathbf{x}) = A\mathbf{x}$ and $\mathbf{b} \in \text{Rng}(f)$, then $\mathbf{b}$ is linearly dependent on the columns of A.

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4. T F If $f(\mathbf{b}) = A^T\mathbf{b}$, then the range of f is the span of the rows of f. In other words, $\text{Rng}(f) = \text{ span } \{[A(i,:)]^T \,:\, i \in \{1, 2, ..., m\}\}$

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5. T F When multiplying A^T and $\mathbf{b}$, it is important to check that the inner dimensions agree.

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6. T F If $A \in \mathbb{R}^{6 \times 5}$, then the function $f(\mathbf{x}) = A\mathbf{x}$ cannot be onto.

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7. T F Matrix-vector multiplication allows us to compute a linear combination of vectors efficiently.

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8. T F Let $A \in \mathbb{R}^{m \times n}$ and suppose $\mathbf{x} \in \mathbb{R}^n$. $A\mathbf{x} = \sum_{k=1}^{n} x_k A(:,k)$

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9. T F Let $A \in \mathbb{R}^{m \times n}$ and suppose $\mathbf{x} \in \mathbb{R}^n$ .If $f(\mathbf{x}) = A\mathbf{x}$, then the codomain of this relation is $\mathbb{R}^n$

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10. T F Let $A \in \mathbb{R}^{m \times n}$ and suppose $\mathbf{x} \in \mathbb{R}^n$. If $f(\mathbf{x}) = A\mathbf{x}$, then the range of f is the span of the columns of f. In other words, $\text{Rng}(f) = \text{ span } \{A(:,k) \,:\, k \in \{1, 2, ..., n\}\} \subseteq \mathbb{R}^m$

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11. T F If $f(\mathbf{b}) = \mathbf{b}^T A$, then the $\text{Rng}(f) = \{\mathbf{x} \in \mathbb{R}^{1 \times n} \,:\, \mathbf{x} = \mathbf{b}^T A \text{ for some } \mathbf{b} \in \mathbb{R}^m\}$

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12. T F Let $A \in \mathbb{R}^{m \times n}$. If $f(\mathbf{x}) = A\mathbf{x}$, then the domain space of this relation is $\mathbb{R}^n$

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13. T F Let $A \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^m$. Then $A^T\mathbf{y} = \sum_{i=1}^{n} y_i \, [A(i,:)]^T$

14. T F For any $A \in \mathbb{R}^{m \times n}$ and $\mathbf{x} \in \mathbb{R}^n$, $A\mathbf{x} = \sum_{k=1}^{n} x_k A(:,k)$.

15. T F Let $A \in \mathbb{R}^{m \times n}$. Define the linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ by $T(\mathbf{x}) = A^T\mathbf{x}$. If $\text{Rng}(T) = \mathbb{R}^n$, then $m \geq n$.

16. T F When multiplying A and $\mathbf{x}$, it is important to check that the inner dimensions must agree.

17. T F If $f(\mathbf{b}) = A^T\mathbf{b}$, then the range of f is the span of the rows of f. In other words, $\text{Rng}(f) = \text{ span } \{[A(i,:)]^T \,:\, i \in \{1, 2, ..., m\}\}$

18. T F If $f(\mathbf{x}) = A\mathbf{x}$ and $\mathbf{b} \in \text{Rng}(f)$, then $\mathbf{b}$ is linearly dependent on the columns of A.

19. T F If $f(\mathbf{x}) = A\mathbf{x}$, then the codomain of this relation is $\mathbb{R}^n$

20. T F If $f(\mathbf{x}) = A\mathbf{x}$, then the range of f is the span of the columns of f. In other words, $\text{Rng}(f) = \text{ span } \{A(:,k) \,:\, k \in \{1, 2, ..., n\}\} \subseteq \mathbb{R}^m$

21. T F $A\mathbf{x} = \sum_{k=1}^{n} x_k A(:,k)$

22. T F Matrix-vector multiplication allows us to compute a linear combination of vectors efficiently.

23. T F Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{x} \in \mathbb{R}^n$. Then the matrix-vector product $A \cdot \mathbf{x}$ represents a linear combination of the rows of A with scalar multiples defined by the entries of $\mathbf{x}$.

Multiple Choice

For the problems below, circle the correct response for each question.

1. Let $A \in \mathbb{R}^{m\times n}$ be given and suppose $\mathbf{x} \in \mathbb{R}^n$. If $f(\mathbf{x}) = A\mathbf{x}$, which of the following statements is false:

   A. $\mathbf{0} \in \text{Rng}(f)$.

   B. The codomain of $f(\mathbf{x})$ is $\mathbb{R}^m$.

   C. The domain of $f(x)$ is $\mathbb{R}^n$.

   D. $\text{Rng}(f) = \{$set of all solutions to the linear-systems problem.$\}$.

   E. $\mathbf{b} \in \text{Rng}(f)$ if and only if $\mathbf{b}$ is a linear combinations of the columns of A.

2. Suppose that $A \in \mathbb{R}^{20\times 5}$ and $\mathbf{x} \in \mathbb{R}^5$. How many total operations on real numbers are necessary to solve the matrix-vector multiplication problem $A\mathbf{x} = \mathbf{b}$? Remember that each multiplication between two real numbers counts as one operation and each addition between two real numbers counts as one operation.

   A. 100 B. 180 C. 90 D. 25 E. 200

3. Let $\mathbf{x} \in \mathbb{R}^2$ be a nonzero vector. Suppose

$$Q = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

   and let $\mathbf{y} = Q\mathbf{x}$. For what angle θ is $\mathbf{x}^T\mathbf{y} = 0$

   A. 0 B. $\frac{\pi}{6}$ C. $\frac{5\pi}{4}$ D. $\frac{4\pi}{3}$ E. $\frac{6\pi}{4}$

4. Define three vectors in $\mathbb{R}^4$ as

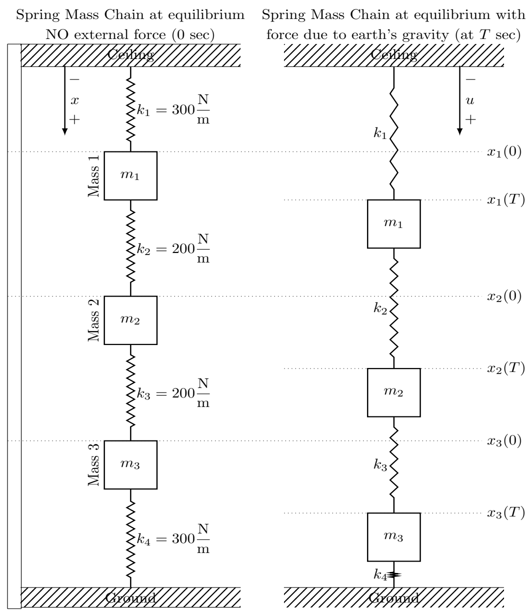
$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 10 \\ 15 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 25 \\ 100 \\ 225 \end{bmatrix}, \qquad \mathbf{a}_4 = \begin{bmatrix} 5 \\ -45 \\ -45 \\ 5 \end{bmatrix}$$

   Please confirm that $\mathbf{a}_4 = 5 \cdot \mathbf{a}_1 - 15 \cdot \mathbf{a}_2 + 1 \cdot \mathbf{a}_3$. Then, choose the vector $\mathbf{x} \in \mathbb{R}^4$ such that

$$\left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{array}\right] \cdot \mathbf{x} = \mathbf{0}$$

   A. $\begin{bmatrix} 45 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ B. $\begin{bmatrix} 5 \\ -15 \\ 1 \\ 1 \end{bmatrix}$ C. $\begin{bmatrix} -5 \\ 15 \\ -1 \\ -1 \end{bmatrix}$ D. $\begin{bmatrix} 5 \\ -15 \\ 1 \\ -1 \end{bmatrix}$ E. The product will never be zero

For the following two problems, consider the following spring-mass system



5. Consider the mass-spring chain from the diagram above. Recall the model for the mass spring chain is given by $M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{F}_e(t)$. Identify the stiffness matrix K for the given values of k_i?

A. $\begin{bmatrix} 500 & -200 & 0 \\ -200 & 400 & -200 \\ 0 & -200 & 500 \end{bmatrix}$ B. $\begin{bmatrix} 300 & -200 & 0 \\ -200 & 200 & -200 \\ 0 & -200 & 300 \end{bmatrix}$ C. $\begin{bmatrix} 500 & -200 & -200 \\ -200 & 400 & -200 \\ -200 & -200 & 500 \end{bmatrix}$

D. $\begin{bmatrix} -500 & 200 & 0 \\ 200 & -400 & 200 \\ 0 & 200 & -500 \end{bmatrix}$ E. $\begin{bmatrix} 300 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}$

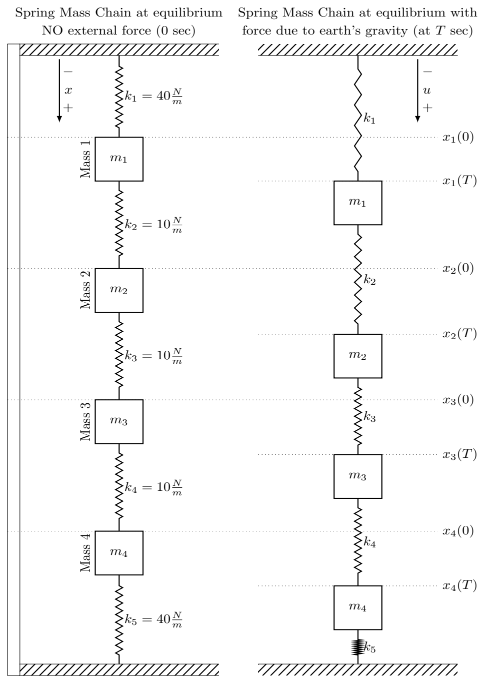
6. Suppose that you are given the displacement vector when $t = T$ at equilibrium to find

$$\mathbf{u} = \begin{bmatrix} u_1(T) \\ u_2(T) \\ u_3(T) \end{bmatrix} = \begin{bmatrix} 0.98 \\ 1.96 \\ 0.98 \end{bmatrix}$$

measured in meters. Then, which of the following gives the mass vector $\mathbf{m} = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix}^T$ as measured in kg? Assume the acceleration due to earth's gravity is $g = 9.8m/s^2$. Also assume that the mass of each spring is zero and that these springs satisfy Hooke's law exactly.

A. $\begin{bmatrix} 10 \\ 40 \\ 10 \end{bmatrix}$ B. $\begin{bmatrix} 9.8 \\ 39.2 \\ 9.8 \end{bmatrix}$ C. $\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ D. $\begin{bmatrix} 0.1 \\ 0.4 \\ 0.1 \end{bmatrix}$ E. $\begin{bmatrix} 29.4 \\ 39.2 \\ 29.4 \end{bmatrix}$

For the next two problems below, consider the following 4-mass, 5-spring chain presented below.



7. Recall that the initial position vector $\mathbf{x}_0$ and the final position vector $\mathbf{x}(T)$ store the positions, measured in meters, of each mass at equilibrium when $t = 0$ and when $t = T$ respectively. Suppose we measure

$$\mathbf{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} 0.200 \\ 0.400 \\ 0.600 \\ 0.800 \end{bmatrix} \qquad \mathbf{x}(T) = \begin{bmatrix} x_1(T) \\ x_2(T) \\ x_3(T) \\ x_4(T) \end{bmatrix} = \begin{bmatrix} 0.249 \\ 0.498 \\ 0.698 \\ 0.849 \end{bmatrix}$$

where each entry is given in meters. Using this information, which of the following vectors gives the force vector $\mathbf{f}_s$ that encodes the forces stored in each spring in this system?

A. $\begin{bmatrix} 1.960 \\ 0.490 \\ 0.000 \\ -0.490 \\ -1.960 \end{bmatrix}$ B. $\begin{bmatrix} 1.960 \\ 0.980 \\ 0.980 \\ 1.960 \end{bmatrix}$ C. $\begin{bmatrix} 1.470 \\ 0.490 \\ 0.000 \\ 0.490 \\ 1.470 \end{bmatrix}$ D. $\begin{bmatrix} 1.470 \\ 0.490 \\ 0.490 \\ 1.470 \end{bmatrix}$ E. $\begin{bmatrix} 0.049 \\ 0.049 \\ 0.000 \\ -0.049 \\ -0.049 \end{bmatrix}$

8. Under the same assumptions as the problem above, which of the following gives the mass vector

$$\mathbf{m} = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \end{bmatrix}^T$$

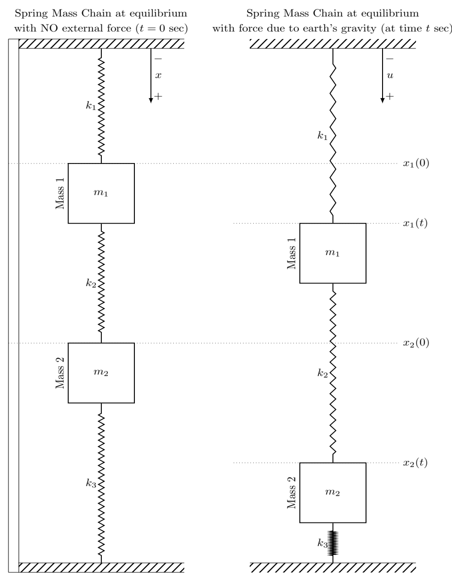
measured in kg, used to produce this position data?

A. $\begin{bmatrix} 0.200 \\ 0.100 \\ 0.100 \\ 0.200 \end{bmatrix}$ B. $\begin{bmatrix} 1.470 \\ 0.490 \\ 0.490 \\ 1.470 \end{bmatrix}$ C. $\begin{bmatrix} 0.250 \\ 0.200 \\ 0.200 \\ 0.250 \end{bmatrix}$ D. $\begin{bmatrix} 2.450 \\ 1.960 \\ 1.960 \\ 2.450 \end{bmatrix}$ E. $\begin{bmatrix} 0.150 \\ 0.050 \\ 0.050 \\ 0.150 \end{bmatrix}$

Free Response

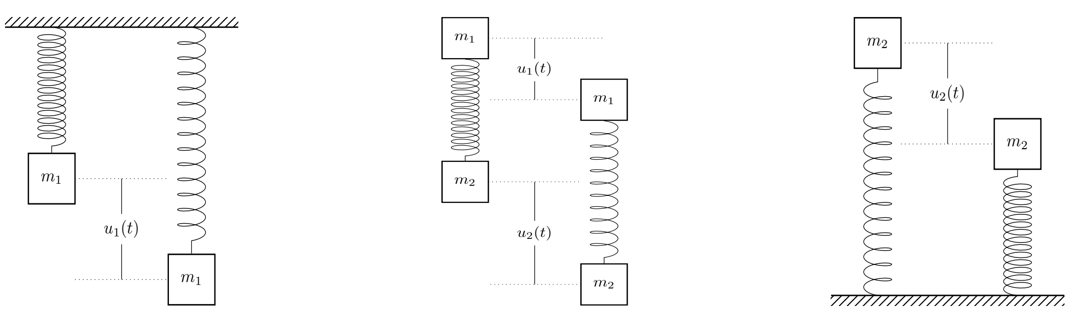
1. Let $m, n \in \mathbb{N}$. Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{x} \in \mathbb{R}^n$. Recall the definition of the column-partition version of matrix-vector multiplication $\mathbf{b} = A \cdot \mathbf{x}$. Also, recall the row-partition version of vector-matrix multiplication $\mathbf{r} = \mathbf{y}^T \cdot A$. How are these definitions related to the linear combination operation? What are the similarities and differences between these two definitions?

For problems 2 - 6 below, consider the following diagram depicting a 2-mass, 3-spring chain with all important variables already labeled.



2. Calculate the displacement vector $\mathbf{u}(t) \in \mathbb{R}^3$ as a linear combination of $\mathbf{x}_0$ and $\mathbf{x}(t)$

3. How much does each spring elongate in this model? Write your answer as a matrix-vector product involving the displacement vector $\mathbf{u}$. Use the diagram below to help you answer this question.



4. Write a matrix-vector product that encodes all Hooke's Law equations in the system. The ith entry of this vector model should state Hooke's Law for the ith spring.

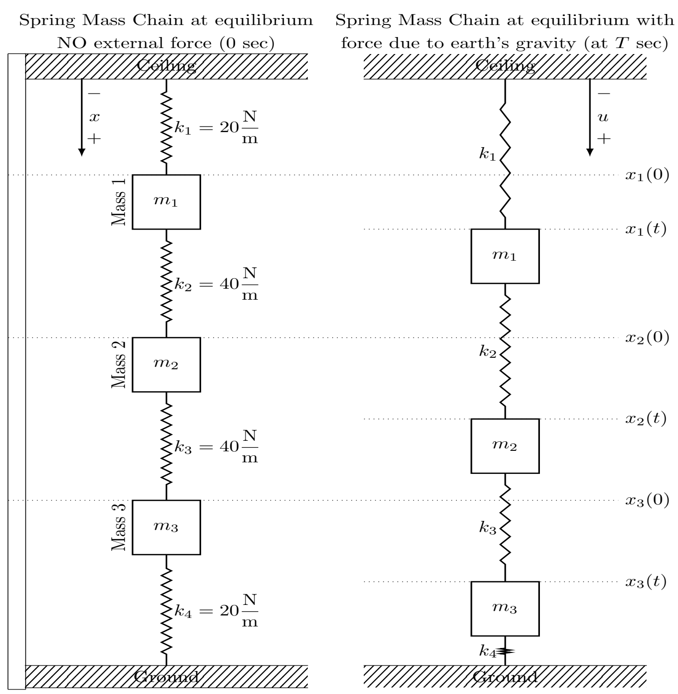
5. Write an vector model for the net forces acting on the system.

6. Use Newton's second law to derive the matrix equation

$$M\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{f}_e$$

where $\mathbf{f}_e$ represents the vector of external forces on each mass. Show the entry-by-entry definition of the mass matrix M.

For problems 7 - 13 below, consider the following diagram depicting a 2-mass, 3-spring chain with all important variables already labeled.



7. Generate vector models (using appropriate matrices and vectors) to define

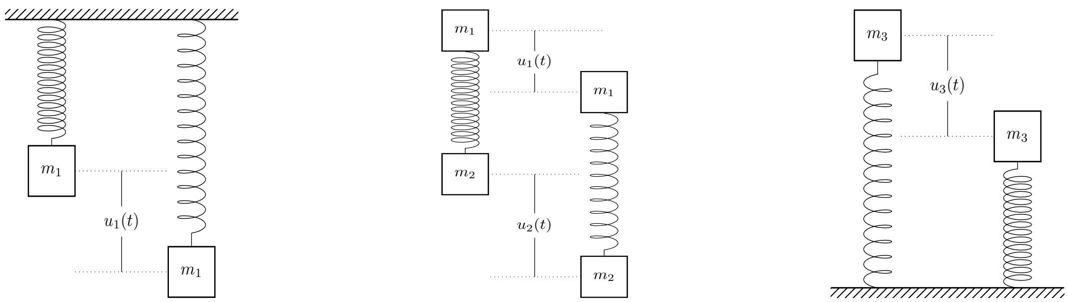
$$\mathbf{x}_0, \mathbf{x}(t), \text{ and } \mathbf{u}$$

where these vectors represent the initial position vector, the final position vector, and the displacement vector, respectively (as discussed in class and in our lesson notes).

8. Show how to calculate the elongation vector $\mathbf{e}$ as a matrix-vector product

$$\mathbf{e} = A\mathbf{u}$$

Write the entry-by-entry definition of matrix A and explain how you derived the equation for each coefficient e_i in this vector. Your answer should include specific references to the diagrams below.



9. Show how to calculate the spring force vector $\mathbf{f}_s$ as a matrix-vector product

$$\mathbf{f}_s = C\mathbf{e}$$

Write the entry-by-entry definition of matrix C and discuss how Hooke's law is used to create the vector of forces for each spring.

10. Create "free-body" diagrams that show all forces acting on each mass m_i. Use these diagrams to derive the vector

$$\mathbf{y} = -A^T\mathbf{f}_s$$

of internal forces. Also, show how to combine your equation for $\mathbf{y}$ with equations from parts B and C to form the stiffness matrix K. Note, you do not have to find the entry-by-entry definition of K.

11. Use Newton's second law to derive the matrix equation

$$M\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{f}_e$$

where $\mathbf{f}_e$ represents the vector of external forces on each mass. Show the entry-by-entry definition of the mass matrix M.

12. Suppose that you are given the displacement vector when $t = T$ at equilibrium to find

$$\mathbf{u} = \begin{bmatrix} u_1(T) \\ u_2(T) \\ u_3(T) \end{bmatrix} = \begin{bmatrix} 0.98 \\ 1.96 \\ 0.98 \end{bmatrix}$$

measured in meters. Find the mass vector $\mathbf{m} = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix}^T$ as measured in kg, that results this displacement? Assume the acceleration due to earth's gravity is $g = 9.8m/s^2$. Also assume that the mass of each spring is zero and that these springs satisfy Hooke's law exactly.

13. Consider the mass-spring chain from the diagram above. Recall the model for the mass spring chain is given by $M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{F}_e(t)$. Calculate the stiffness matrix K for the given values of k_i?