Answers **Lesson 0 Warm Up Quiz**

Math 2B: Applied Linear Algebra

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1. State the applied math modeling process. Explain where matrices and vectors arise in this applied mathematical modeling process.

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2. State the Matrix-Vector Multiplication problem (MVMP). Specifically identify the given information and what is unknown. Make sure to state the dimensions of all quantities in this problem statement

Solution:

Problem 1: The Matrix-Vector Multiplication Problem

Let $m, n \in \mathbb{N}$. Let $A \in \mathbb{R}^{m \times n}$ be a given matrix and $\mathbf{x} \in \mathbb{R}^n$ be a given vector. Then the matrix-vector multiplication problem is to find an unknown vector $\mathbf{b} \in \mathbb{R}^m$ such that

$$A \cdot \mathbf{x} = \mathbf{b}$$

Matrix-vector multiplication is a "forward problem." To understand this terminology, let's define the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $f(\mathbf{x}) = A \cdot \mathbf{x}$. Based on this definition, we will see that function f satisfies the following:

- the domain of f is $\mathbb{R}^n$
- the codomain f is $\mathbb{R}^m$
- the range of f is contained in $\mathbb{R}^m$ but may not be equal to $\mathbb{R}^m$ depending on the entries in the columns of matrix A

In this case, the function f is defined as a matrix-vector product. Any matrix A implicitly defines the matrix-vector product function f, as illustrated above. Matrix-vector multiplication is a forward problem because we start with a given input $\mathbf{x}$ in the domain $\mathbb{R}^n$ and move forward into the range to find the output vector $\mathbf{b}$. When solving the matrix-vector multiplication problem, we map from the domain forward into the range. Hence, we call this a forward problem.

In order to create such a problem, we need to construct a matrix A that models some component of a physical situation. Then, we need to determine an input vector $\mathbf{x}$ representing a known input state from our physical problem. The unknown output $\mathbf{b} \in \mathbb{R}^m$ is a solution to this problem if it can calculated using matrix-vector multiplication.

3. State the Nonsingular Linear-Systems problem (NLSP). Specifically identify the given information and what is unknown. Make sure to state the dimensions of all quantities in this problem statement

Solution:

Problem 2A: The Nonsingular Linear-Systems Problem

Let $n \in \mathbb{N}$. Let $A \in \mathbb{R}^{n \times n}$ be a given nonsingular matrix and $\mathbf{b} \in \mathbb{R}^n$ be a given vector. Then the nonsingular linear-systems problem is to find an unknown vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$A \cdot \mathbf{x} = \mathbf{b}.$$

Just like matrix-vector multiplication, we can describe the nonsingular linear-systems problem using the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined as $f(\mathbf{x}) = A \cdot \mathbf{x}$. Based on this definition, we have:

- the domain of f is $\mathbb{R}^n$
- the codomain f is $\mathbb{R}^n$.
- the range of f is $\mathbb{R}^n$ due to the structure of the nonsingular matrix A

The nonsingular linear-systems problem is a backward problem because we start with the function description, defined by the nonsingular matrix A, and we are given one specific output vector $\mathbf{b}$ in the range of the function f. From this information, we are asked to find all possible vectors $\mathbf{x}$ in the domain of our function such that

$$f(\mathbf{x}) = \mathbf{b}.$$

When solving the linear-systems problem, we begin in the range and work our way backwards into the domain. Hence, we call this a backward, or inverse, problem.

To craft a nonsingular linear-systems problem, we need to construct a square, nonsingular matrix A that describes a physical phenomenon. As we will see, there are many real-world applications that result in these special types of matrices. With our nonsingular matrix A in hand, we also need to determine a vector $\mathbf{b}$ that represents an output state in our system. We construct our model in such a way that output vector $\mathbf{b}$ can be written as the product of our matrix A with some unknown input vector $\mathbf{x} \in \mathbb{R}^n$. The solution to the nonsingular linear-systems problem is any input vector that results in output $\mathbf{b}$ after a matrix-vector multiplication with matrix A.

4. How are the MVMP and the NSLP problems related? How are these problems similar? How do these problems different?

Solution: A major similarity between matrix-vector multiplication and the nonsingular linear-systems problem is that both depend on matrix-vector multiplication. The process of solving the former problem is simply to calculate a product. To solve the later problem, we "reverse engineer" our matrix-vector product in order to find input vectors that produce a given output. However, both problems involve the same underlying matrix-vector multiplication function

$$f(\mathbf{x}) = A \cdot \mathbf{x}.$$

One of the major differences between the matrix-vector multiplication problem and the nonsingular linear-systems problem is in the assumptions on the matrix A. For general matrix-vector products, we need only create a rectangular $m \times n$ matrix, where m may not be equal to n. No additional structure on the matrix A is needed in order to solve this problem. On the other hand, for the nonsingular linear-systems problem, we need to create a matrix with the same number of rows as columns and this matrix must have very special column structure.

5. State the General Linear-Systems problem (GLSP). Specifically identify the given information and what is unknown. Make sure to state the dimensions of all quantities in this problem statement

6. State the Full-Rank Least-Squares problem (FRLSP). Specifically identify the given information and what is unknown. Make sure to state the dimensions of all quantities in this problem statement

Solution: The least-squares problem is as follows:

> Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $\mathbf{b} \in \mathbb{R}^m$, find all unknown vectors $\mathbf{x} \in \mathbb{R}^n$ that minimize the norm of the residual vector $\mathbf{b} - A\mathbf{x}$. In other words, find
>
> $$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{b} - A\mathbf{x}\|_2$$

Just like the matrix-vector multiplication and linear-systems problems, we contextualize the least-squares problem using function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad f(\mathbf{x}) = A\mathbf{x} = \sum_{k=1}^{n} x_k\, A(:,k)$$

In this case, we see:

- the domain of f is $\mathbb{R}^n$
- the codomain f is $\mathbb{R}^m$.
- the range of f is $\text{Span}\{A(:,k)\}_{k=1}^{n}$

The least-squares problem is a backward problem because we start with the function description (defined by matrix A) and we are given one specific output value $\mathbf{b}$ in the CODOMAIN of $f(\mathbf{x})$.

From this information, we are asked to find all possible input values $\mathbf{x}$ in the domain that product output value$f(\mathbf{x})$ "as close as possible" the the vector $\mathbf{b}$. When solving the least-squares problem, we begin in the codomain and work our way backwards to the domain. Hence, we call this a backward problem.

Using our definition of the least-squares problem, we can classify all linear-systems problems as least-squares problems. For linear systems problems, since $\mathbf{b}$ begins in the range of $f(\mathbf{x})$, we know that

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{b} - A\mathbf{x}\|_2 = 0$$

for each solution. However, not all least-squares problems are linear systems problems. In the case that we have a $\mathbf{b} \in \mathbb{R}^m$ with $\mathbf{b} \notin \mathrm{Rng}(f)$, we know that $f(\mathbf{x}) \neq \mathbf{b}$ for all $\mathbf{x} \in \mathbb{R}^n$. This is the most interesting case and is the focus of our discussion of the least-squares problem. Thus, the least-squares is a generalization of the linear systems problem that enables us to create "best-fit" approximations to noisy data.

7. How are the GLSP and the FRLSP problems related? How are these problems similar? How do these problems different?

8. State the Standard Eigenvalue problem (SEP). Specifically identify the given information and what is unknown. Make sure to state the dimensions of all quantities in this problem statement