Name: Lesson 12 Solutions Class #: ∞

LESSON 12: Radical Expressions, Functions, and Models

- ☐ $\sqrt{a} = \sqrt[2]{a}$: Square root of a
- ☐ Radical sign, index and radicand
- ☐ Calculating roots using calculator
- ☐ Simplifying $\sqrt{(a)^2}$ using the absolute value
- ☐ $\sqrt[3]{a}$: Cube root of a
- ☐ $\sqrt[n]{a}$: the nth root of a for odd index n
- ☐ $\sqrt[n]{a}$: the nth root of a for even index n

Anatomy of a pure power

constant power

variable base b whose value will be given

$$b^n = a$$

← unknown output of a power

For each of the following power expressions, do each of the following:

i. Specifically identify the value of base b and the value of power n

iii. Evaluate the expression

The first one is done for you.

1A. 11^2 "eleven squared"

base $b = 11$
power $n = 2$

$a = 11^2$

$= 11 \cdot 11$ ← base $b = 11$ multiplied by itself $n = 2$ times

$= \boxed{121}$

1B. 2^6 ← "two to the sixth power"

base $b = 2$
power $n = 6$

$a = 2^6$

$= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ ← base $b = 2$ multip[lied] by itself $n = 6$ times

$= \boxed{64}$

1C. 3^4 ← "three to the fourth power"

base $b = 3$
power $n = 4$

$a = 3^4$

$= 3 \cdot 3 \cdot 3 \cdot 3$ ← base $b = 3$ multiplied by itself $n = 4$ times

$= \boxed{81}$

1D. 5^3 ← "five to the third power"

base $b = 5$
power $n = 3$

$a = 5^3$

$= 5 \cdot 5 \cdot 5$ ← base $b = 5$ multip[lied] by itself $n = 3$ tim[es]

$= \boxed{125}$

radicals are inverse power problems

Name: Lesson 12 Solutions

Class #: ∞

Backward Problem: anatomy of radicals

index

radical symbol

unknown variable base $\rightarrow b = \sqrt[n]{a} \iff b^n = a$

radicand : known output of a power

For each of the following power expressions, do each of the following:

i. Specifically identify the value of index n and the value of radicand a

iii. Evaluate the expression by transforming each expression into a power equation

The first one is done for you.

2A. $\sqrt[2]{100}$ ← "the second root of one hundred" or "the square root of 100"

$\begin{cases} \text{index } n = 2 \\ \text{radicand } a = 100 \end{cases}$

$b = \sqrt[2]{100} \iff b^2 = 100$

- we are looking for base b such that $b^2 = b \cdot b = 100$.
- we have two choices: either $b = +10$ or $b = -10$.
- For even index, we always choose the positive to avoid confusion

$\Rightarrow \boxed{b = +10}$

since $10^2 = 100$

2B. $\sqrt[3]{27}$ ← "the third root of 27"

$\begin{cases} \text{index } n = 3 \\ \text{radicand } a = 27 \end{cases}$

$b = \sqrt[3]{27} \iff b^3 = 27$

- we are looking for a base b such that $b^3 = b \cdot b \cdot b = 27$.
- Here we have only one choice of $b = 3$
- When we have an odd index, we will only have one choice.

$\Rightarrow \boxed{b = 3}$

since $3^3 = 27$

2C. $\sqrt[5]{32}$ ← "the fifth root of 32"

$\begin{cases} \text{index } n = 5 \\ \text{radicand } a = 32 \end{cases}$

$b = \sqrt[5]{32} \iff b^5 = 32$

- we are looking for base b such that $b^5 = b \cdot b \cdot b \cdot b \cdot b = 32$
- Once again, with an odd index, we have only one option of $b = 2$

$\Rightarrow \boxed{b = 2}$ since $2^5 = 32$

2D. $\sqrt[4]{81}$ ← "the fourth root of 81"

$\begin{cases} \text{index } n = 4 \\ \text{radicand } a = 81 \end{cases}$

$b = \sqrt[4]{81} \iff b^4 = 81$

- we're looking for base b such that $b^4 = b \cdot b \cdot b \cdot b = 81$
- Since the index is even, we have two possible choices of either $b = +3$ or $b = -3$. resulting from the facts that
  - positive x positive = positive
  - negative x negative = positive
- To avoid confusion, we always choose the positive choice

$\Rightarrow \boxed{b = +3}$

since $3^4 = 81$

Name: Lesson 12 Solutions Class #: ∞

3. Evaluate each entry of the tables below. Then, in the last row of the table, specifically identify the index of each radical expression.

TABLE 3A: Values of $\sqrt[2]{x^2}$	
Input x	Output $y = \sqrt[2]{x^2}$
-3	$\sqrt[2]{(-3)^2} = \sqrt[2]{9} = +3$ *
-2	$\sqrt[2]{(-2)^2} = \sqrt[2]{4} = +2$
-1	$\sqrt[2]{(-1)^2} = \sqrt[2]{1} = +1$
0	$\sqrt[2]{0^2} = \sqrt[2]{0} = 0$
1	$\sqrt[2]{1^2} = \sqrt[2]{1} = 1$
2	$\sqrt[2]{2^2} = \sqrt[2]{4} = 2$
3	$\sqrt[2]{3^2} = \sqrt[2]{9} = 3$ *
What is the index of $y = \sqrt[2]{x^2}$: $n=2$ (even)	

TABLE 3B: Values of $\sqrt[3]{x^3}$	
Input x	Output $y = \sqrt[3]{x^3}$
-3	$\sqrt[3]{(-3)^3} = \sqrt[3]{-27} = -3$
-2	$\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$
-1	$\sqrt[3]{(-1)^3} = \sqrt[3]{-1} = -1$
0	$\sqrt[3]{0^3} = \sqrt[3]{0} = 0$
1	$\sqrt[3]{1^3} = \sqrt[3]{1} = 1$
2	$\sqrt[3]{2^3} = \sqrt[3]{8} = 2$
3	$\sqrt[3]{3^3} = \sqrt[3]{27} = 3$
What is the index of $y = \sqrt[3]{x^3}$: $n=3$ (odd)	

* Recall: the radical $\sqrt[2]{9} = b \Leftrightarrow b^2 = 9$ yields two possible options of $b=-3$ or $b=+3$. By convention, we agree to choose $b=+3$ (positive) to avoid confusion.

4. Look at the output values of $y = \sqrt[2]{x^2}$ in table 3A. What pattern do you notice about these output values versus the input values of x? Why do the negative signs on the input values of x "disappear" in this table? What function behaves like this? This function behaves like the absolute value function. The output values of $\sqrt[2]{x^2}$ are all positive. This operation transforms negative inputs into their positive version and leaves positive inputs as positive values. This transformation results from two operations. First, when we multiply a negative input by itself two times, we get a positive radicand. Next, when we take the square root of a positive number, we agree to take the positive base that, when squared, produces the radical.

5. Look at the output values of $y = \sqrt[3]{x^3}$ in tables 3B. What pattern do you notice about these output values versus the input values of x? Why DON'T the negative input values of x "disappear" in this table? In Table 3B, each output is identical to each corresponding input. Negative inputs remain negative in the output and positive inputs remain positive. The operation $\sqrt[3]{x^3}$ seems not to change the value of our input. This tendancy of maintaining the sign and magnitude of each input results from the fact that a negative x negative x negative = negative, positive x positive x positive = positive and thus the radical produces only one unique choice.

Name: Lesson 12 Solutions Class #: ∞

6. Evaluate each entry of the tables below. Then, in the last row of the table, specifically identify the index of each radical expression.

TABLE 3C: Values of $\sqrt[4]{x^4}$	
Input x	Output $y = \sqrt[4]{x^4}$
-2	$\sqrt[4]{(-2)^4} = \sqrt[4]{16} = +2$ *
-1	$\sqrt[4]{(-1)^4} = \sqrt[4]{1} = +1$
0	$\sqrt[4]{0^4} = \sqrt[4]{0} = 0$
1	$\sqrt[4]{1^4} = \sqrt[4]{1} = 1$
2	$\sqrt[4]{2^4} = \sqrt[4]{16} = 2$ *
What is the index of $y = \sqrt[4]{x^4}$: $n = 4$ (even)	

TABLE 3D: Values of $\sqrt[5]{x^5}$	
Input x	Output $y = \sqrt[5]{x^5}$
-2	$\sqrt[5]{(-2)^5} = \sqrt[5]{-32} = -2$
-1	$\sqrt[5]{(-1)^5} = \sqrt[5]{-1} = -1$
0	$\sqrt[5]{0^5} = \sqrt[5]{0} = 0$
1	$\sqrt[5]{1^5} = \sqrt[5]{1} = 1$
2	$\sqrt[5]{2^5} = \sqrt[5]{32} = 2$
What is the index of $y = \sqrt[5]{x^5}$: $n = 5$ (odd)	

* Recall: the radical $b = \sqrt[4]{16} \Leftrightarrow b^4 = 16$ yields two possible options of $b = -2$ or $b = +2$. Again, by convention, we agree $b = +2 = \sqrt[4]{16}$ (positive) to avoid confusion. This choice implies negative input are mapped to positive outputs when index n is even.

7. Look at the output values of $y = \sqrt[4]{x^4}$ in table 3C. What pattern do you notice about these output values versus the input values of x? Why do the negative signs on the input values of x "disappear" in this table? What function behaves like this? Just like in table 3A, the output values of $y = \sqrt[4]{x^4}$ behave like the absolute value function $|x| = \sqrt[4]{x^4}$. In particular, the $\sqrt[4]{x^4}$ operation transforms negative input values into the corresponding positive value while leaving positive input values as positive in the output. This is exactly what the absolute value function does. Again, this pattern results from a more general observation that [positive x positive x positive x positive = positive; negative x negative x negative x negative = positive]. When we take the fourth root, we choose the positive base that generates the radicand.

8. Look at the output values of $y = \sqrt[5]{x^5}$ in tables 3D. What pattern do you notice about these output values versus the input values of x? Why DON'T the negative input values of x "disappear" in this table?

Just like in table 3B, the output values of $\sqrt[5]{x^5}$ are identical to the corresponding input values x. In particular, negative inputs produce identical, negative outputs while positive inputs produce identical positive outputs. This results from facts:

positive x positive x positive x positive x positive = positive
negative x negative x negative x negative x negative = negative

Thus, when we attempt to invert each resulting radicand using a fifth root, we are left with a unique choice.

Name: Lesson 12 Solutions Class #: ∞

INVERSE OPERATIONS FOR ODD POWERS

Suppose index $n = 3, 5, 7, 9, \ldots$ is an odd number

$$\sqrt[n]{x^n} = x$$

- When applying a radical operation to invert a power operation, radicals with odd index are "pure" inverses that annihilate the power leaving the base x with no special considerations. This is a dream come true since we can get rid of odd powers with odd radicals

- For example,
  - $\sqrt[3]{x^3} = x$
  - $\sqrt[5]{y^5} = y$
  - $\sqrt[7]{d^7} = d$

← odd-indexed radicals annihilate odd powers exactly (pure inverse)

INVERSE OPERATIONS FOR EVEN POWERS

Suppose index $n = 2, 4, 6, 8, \ldots$ is an even number:

$$\sqrt[n]{x^n} = |x|$$

- When applying a radical operation to invert a power operation, radicals with even index are "special" inverses that require extra thought. In particular, even-indexed radicals do not annihilate the even power unless we include absolute value bars. Thus, when using even-indexed radicals to annihilate even powers, we produce an absolute value expression that results from our inverse. This more nuanced inverse property is a generalization from tables 3A and 3C above. Remember even-indexed radicals invert even powers to produce absolute value expressions.

- For example:
  - $\sqrt[2]{a^2} = |a|$
  - $\sqrt[4]{x^4} = |x|$
  - $\sqrt[6]{z^6} = |z|$

Note: In these problems, we want to get each expression into a form identical to our inverse formulas on page 5 of this handout. Thus, we need pure powers in the radicand.

Name: Lesson 12 Solutions   Class #: ∞

Simplify each expression below using the rules for radicals with an even and radicals with an odd index

6A. $\sqrt[2]{w^2}$ ← this radicand is already a perfect power. Thus, we can apply our even-indexed radical inverse formula

$$\Rightarrow \sqrt[2]{w^2} = \boxed{|w|}$$

6B. $\sqrt[4]{16 \cdot b^4}$ ← this radicand is not written as a perfect power. Let's try to manipulate this a bit.

Note: $16 \cdot b^4 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot b \cdot b \cdot b \cdot b$

$= (2 \cdot b) \cdot (2 \cdot b) \cdot (2 \cdot b) \cdot (2 \cdot b)$

$= (2 \cdot b)^4$

$$\Rightarrow \sqrt[4]{16 \cdot b^4} = \underbrace{\sqrt[4]{(2 \cdot b)^4}}$$

this is the exact form of our even-indexed inverse formula (i.e. use abs value)

$= |2 \cdot b|$

$= \boxed{2 \cdot |b|}$

6C. $\sqrt[5]{32 \cdot a^{10}}$ ← this radicand is not written as a perfect fifth power. Let's do some math.

Note: $32 \cdot a^{10} = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot a^2 \cdot a^2 \cdot a^2 \cdot a^2 \cdot a^2$

$= (2 \cdot a^2) \cdot (2 \cdot a^2) \cdot (2 \cdot a^2) \cdot (2 \cdot a^2) \cdot (2 \cdot a^2)$

$= (2 \cdot a^2)^5$

$$\Rightarrow \sqrt[5]{32a^{10}} = \sqrt[5]{(2 \cdot a^2)^5}$$ ← this is the exact form of the odd-indexed inverse formula and thus we get a pure cancel

$= \boxed{2 \cdot a^2}$

6B. $\sqrt[3]{-125y^3}$ ← this radicand is not a perfect third power as written

Note: $-125 \cdot y^3 = -5 \cdot -5 \cdot -5 \cdot y \cdot y \cdot y$

$= (-5 \cdot y) \cdot (-5 \cdot y) \cdot (-5 \cdot y)$

$= (-5 \cdot y)^3$

$$\Rightarrow \sqrt[3]{-125y^3} = \sqrt[3]{(-5 \cdot y)^3}$$

$= \boxed{-5 \cdot y}$