Lesson 8: Intro to Numerical Analysis

Announcements:

- [x] Lesson 6, part 1 & 2 notes and Lesson 7 notes have been posted for ≥ 7 days
- [ ] In-class exam on Wed 3/20/2019 from 8am - 9:50am (with the option to start at 7am)
- [ ] In-class exam will focus on all content through the end of todays Lesson (hopefully including $\boxed{\text{Newton's Method}}$)
- [ ] Write your own exam question assignment due on Mon 3/18/2019 @ 8am
- [ ] Lab 5 on square roots will be due on or before Thurs 3/28/2019 @ 8am in my office
- [ ] If you expect to earn a grade for your in-class final, then you should be in class on Mon 3/25/2019 from 8am - 10am for regularly scheduled final exam

Trefethen's Definition

Numerical Analysis is the study of algorithm for the problems of continuous mathematics.

Numerical Analysis involves

- the study of **algorithms**
  - A. design algorithms (make it work)
  - B. analyze algorithms (verify & certify)

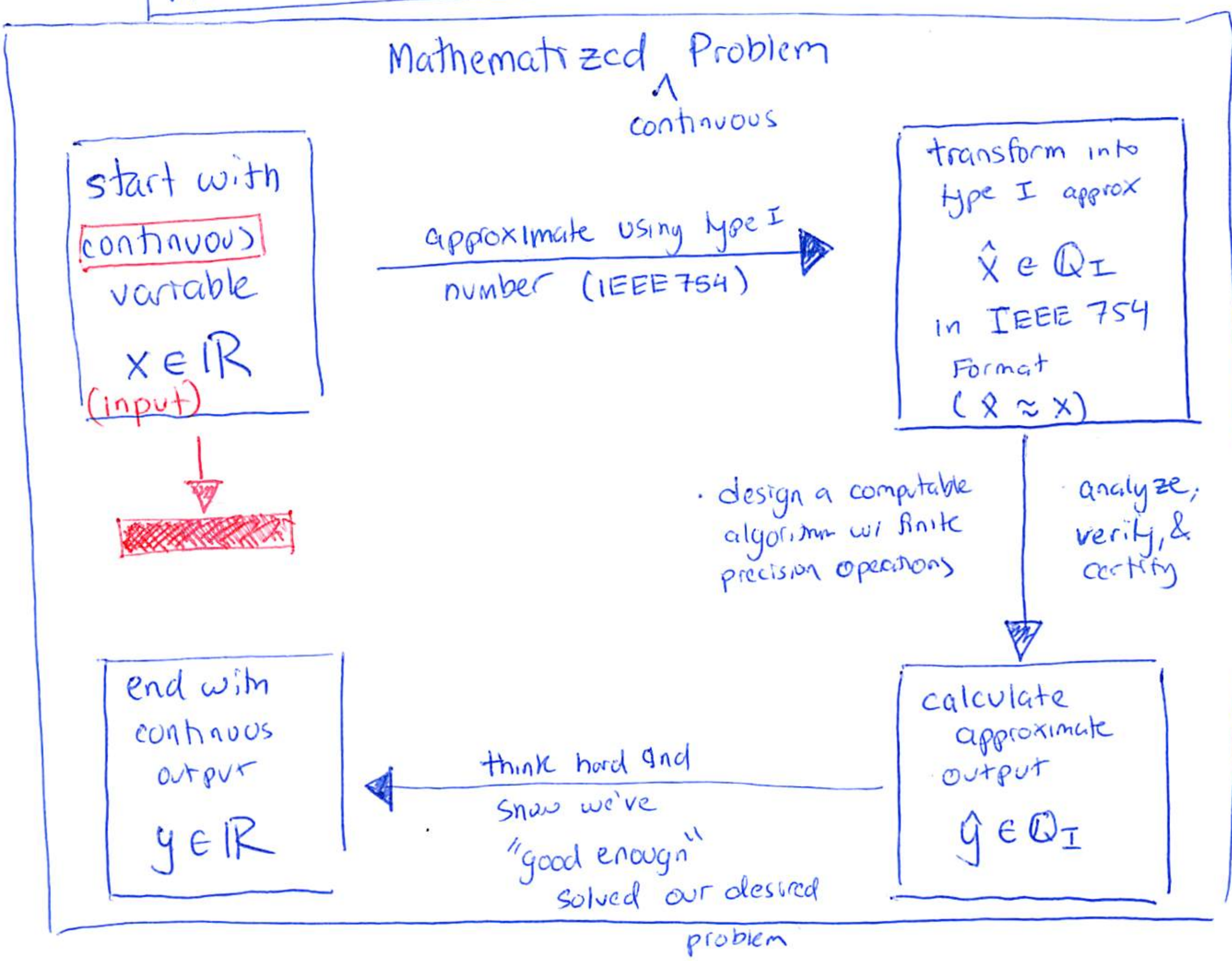
- Problems of **continuous mathematics**
  - A. stated with real or complex variables (type II)
  - B. If we want to store and operate on $x \in \mathbb{R}$ on a computer, we must use $\hat{x} \in \mathbb{Q}_I$
    (all work we do will be an approximation !!)
  - C. continuous is opposite of discrete
    (discrete problems are solved in other fields of computer science)

☐ work that is applied

A. this is a major engine behind STEM

☐ Approximating unknowns

Visual description



(the errors are tolerable)

Let's focus on equation solving:

Lets look at our input $x \in \mathbb{R}$

and let's try to calculate

$$y = \sqrt{x} \in \mathbb{R}$$

eg: Let $x = 2$. Claim $\sqrt{2} \in \mathbb{R}_{II}$ (type II real number, irrational)

$$y = \sqrt{2}$$ ← let's do some transformation (math analysis as a pre-step to algorithm development)

$$\Rightarrow \quad y^2 = 2$$

$$\Rightarrow \quad \underbrace{y^2 - 2} = 0$$

$$\Rightarrow \quad f(y) = 0$$

input

Given an $x \in \mathbb{R}$, find the "roots" of a differentiable function $f(y)$ s.t. $f(y) = 0$

output of algorithm

Def: Root Problem (Equation solving)

Find the "roots" of function $f(y)$

aka $\boxed{\text{find the } y \text{ values}}$ s.t.

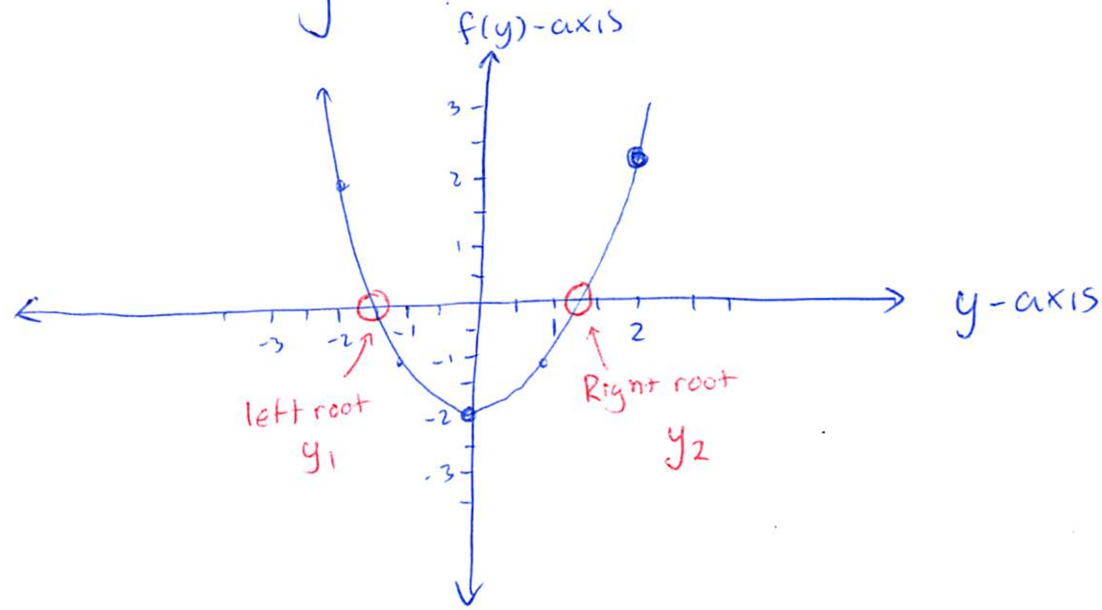
$$f(y) = 0$$

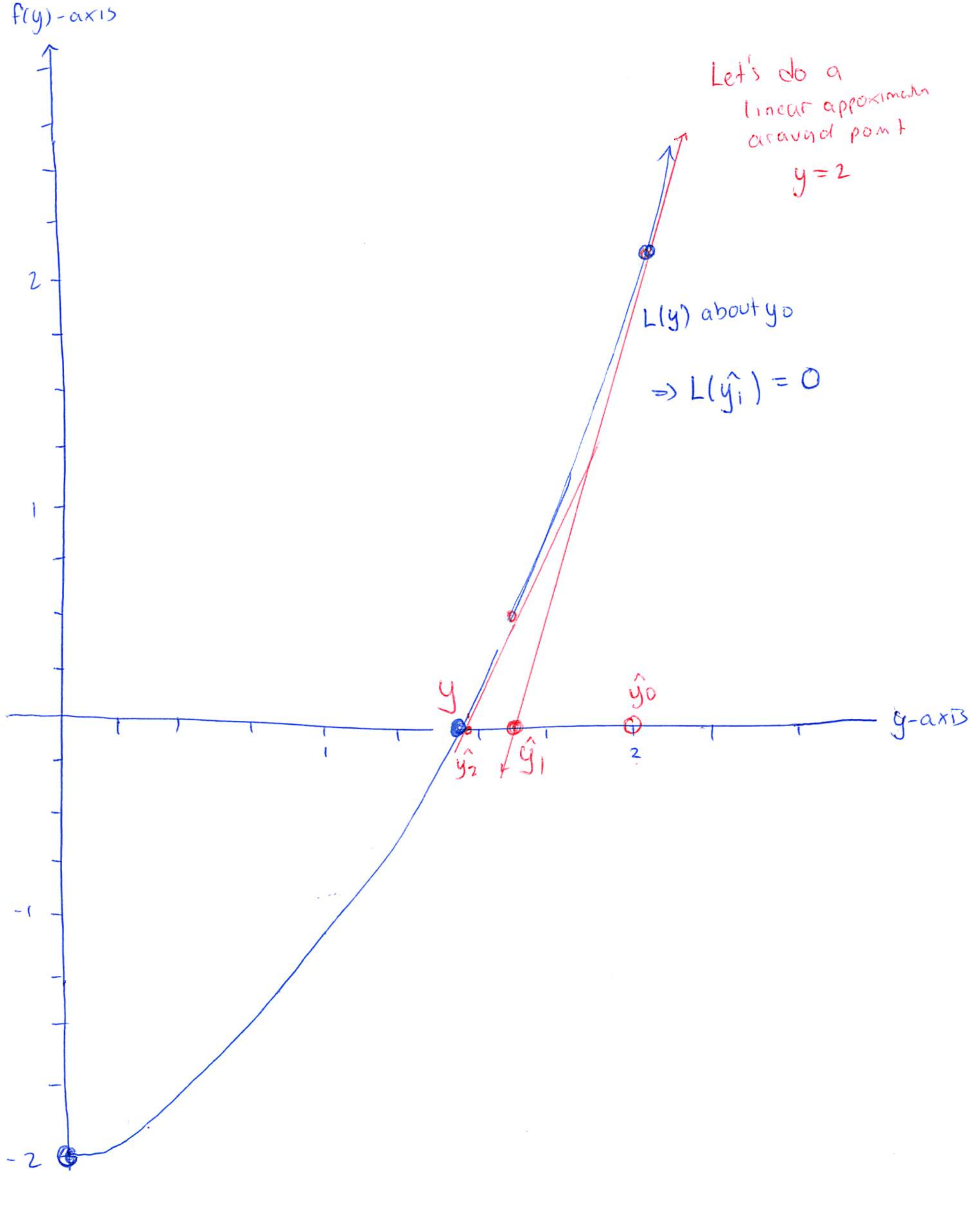
"produce y-values as output of algorithm"

eg: If $x = 2$, the WTF y s.t.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$f(y) = y^2 - 2 = 0$$




f(y)-axis
Let's do a linear approximation around point y = 2
2
L(y) about y0
⇒ L(ŷ1) = 0
1
y
ŷ0
y-axis
1
ŷ2
ŷ1
2
-1
-2

point-slope
equation for lines

For point (x_1, y_1) and slope m

the equation for a line is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y = y_1 + m(x - x_1)$$

linear approx

$$\Rightarrow \quad L(x) = f(a) + f'(a) \cdot (x - a)$$

tangent
line at
point
$(a, f(a))$
$\Rightarrow m = f'(a)$

$a = y_0 = 2 \quad , \quad x = y$

$$L(y) = f(y_0) + f'(y_0)(y - y_0)$$

$$\Rightarrow \quad L(\boxed{y}) = f(2) + f'(2)(\boxed{y} - 2)$$

$y = \hat{y}_1 = 0$

$f(y) = y^2 - 2$

$\Rightarrow f(2) = 2^2 - 2 = 2$

$\Rightarrow f'(y) = 2y$

$\Rightarrow f'(2) = 4$

$$L(\hat{y_i}) = 0 = f(y_0) + f'(y_0)\,(\boxed{y} - y_0)$$

WTF

$$f(y) = y^2 - 2$$

$$\Rightarrow f'(y) = 2y$$

$$\Rightarrow f'(y_0) = 2 \cdot y_0$$

$$\Rightarrow \quad 0 = f(y_0) + f'(y_0) \cdot (y - y_0)$$

$\hat{y_0}$ constant

$\Rightarrow f(\hat{y_0})$ constant

$\Rightarrow f'(\hat{y_0})$ constant

$\Rightarrow$

$\Rightarrow$

$$0 = f(y_0) + f'(y_0)\cdot(y - y_0)$$

$$\Rightarrow -f(y_0) = f'(y_0)(y - y_0)$$

$$\Rightarrow \frac{-f(y_0)}{f'(y_0)} = y - y_0$$

$$0 = 5 + 2(y-3)$$

$$\Rightarrow -5 = 2(y-3)$$

$$\Rightarrow \frac{-5}{2} = y - 3$$

$$\Rightarrow y = 3 + \frac{-5}{2}$$

$$\Rightarrow \boxed{y} = \hat{y}_0 + \frac{-f(\hat{y}_0)}{f'(\hat{y}_0)}$$

Algorithm design

Newton's Method

$$\Rightarrow \boxed{\hat{y}_1 = \hat{y}_0 - \frac{f(\hat{y}_0)}{f'(\hat{y}_0)}}$$

$$\hat{y}_2 = \hat{y}_1 - \frac{f(\hat{y}_1)}{f'(\hat{y}_1)}$$

$\vdots$

$$\boxed{\hat{y}_{n+1} = \hat{y}_n - \frac{f(\hat{y}_n)}{f'(\hat{y}_n)}}$$

$$f(y) = y^2 - 2$$

$$f'(y) = 2y$$

$$f(\hat{y}_n) = \hat{y}_n^2 - 2$$

$$f'(\hat{y}_n) = 2\cdot\hat{y}_n$$

$$\hat{y}_{n+1} = \frac{\hat{y}_n}{1} - \frac{(\hat{y}_n^2 - 2)}{2 \cdot \hat{y}_n}$$

$$= \frac{2\hat{y}_n^2 - (\hat{y}_n^2 - 2)}{2\hat{y}_n}$$

$$= \frac{\hat{y}_n^2 + 2}{2\hat{y}_n}$$

$$= \frac{1}{2}\left(\frac{\hat{y}_n^2 + 2}{\hat{y}_n}\right)$$

$$= \frac{1}{2}\left(\frac{\hat{y}_n^2}{\hat{y}_n} + \frac{2}{\hat{y}_n}\right)$$

$$\boxed{\hat{y}_{n+1} = \frac{1}{2}\left(\hat{y}_n + \frac{2}{\hat{y}_n}\right)}$$

$$\frac{a}{1} - \frac{(a^2-2)}{2 \cdot a}$$

$$\frac{2a^2}{2a} - \frac{(a^2-2)}{2a}$$

$$\frac{a^2+2}{2a}$$

$$\frac{1}{2}\left(\frac{a^2+2}{a}\right)$$

$$\frac{1}{2}\left(\frac{a^2}{a} + \frac{2}{a}\right)$$

$$\frac{1}{2}\left(a + \frac{2}{a}\right)$$

$y^2 - 11 = 0$ $\qquad$ $\hat{y}_{n+1} = \frac{1}{2}\left(\hat{y}_n + \frac{11}{\hat{y}_n}\right)$

Let $f(y) = y^2 - x$

output

input
(the number your trying to root)

Let
$$\boxed{\begin{aligned} y_0 &= \hat{x} \approx x = 2 \\ y_{n+1} &= \boxed{\frac{1}{2}}\left(\hat{y_n} + \frac{\hat{x}}{\hat{y_n}} \right) \end{aligned}}$$