Monday 1/25/2021 @ 11am

Want to prove : $\boxed{\sum_{K=1}^{n} K^2} = \boxed{\frac{n(n+1)(2n+1)}{6}}$

LHS RHS

Let's start by building our understanding.

$n = 1$

$$\text{LHS} : \sum_{K=1}^{n} K^2 = \sum_{K=1}^{1} K^2$$

$$= 1^2$$

$$= 1$$

RHS: $$\left.\frac{n(n+1)(2n+1)}{6}\right|_{n=1} = \frac{1\cdot(1+1)(2\cdot 1+1)}{6}$$

$$= \frac{1\cdot 2\cdot 3}{6}$$

$$= \frac{6}{6}$$

$$= 1 \checkmark$$

$n = 2$

$$\text{RHS} = \left.\frac{n \cdot (n+1) \cdot (2n+1)}{6}\right|_{n=2}$$

$$= \frac{2 \cdot (2+1)(2 \cdot 2+1)}{6}$$

$$= \frac{2 \cdot 3 \cdot 5}{6}$$

$$= \frac{6 \cdot 5}{6}$$

$$= 5 \ \checkmark$$

$n = 2$

$$\text{LHS} = \sum_{K=1}^{n} K^2$$

$$= \sum_{K=1}^{2} K^2$$

$$= \underset{(K=1)}{1^2} + \underset{(K=2)}{2^2}$$

$$= 1 + 4$$

$$= 5 \checkmark$$

$n = 3$

$$LHS = \sum_{K=1}^{n} K^2$$

$$= \sum_{K=1}^{3} K^2$$

$$= \underset{(K=1)}{1^2} + \underset{(K=2)}{2^2} + \underset{(K=3)}{3^2}$$

$$= 1 + 4 + 9$$

$$= 14 \checkmark$$

$n=3$

$$RHS = \left.\frac{n\cdot(n+1)\cdot(2n+1)}{6}\right|_{n=3}$$

$$= \frac{3\cdot(3+1)\cdot(2\cdot 3+1)}{6}$$

$$= \frac{3\cdot 4\cdot 7}{6}$$

$$= \frac{12\cdot 7}{6}$$

$$= 2\cdot 7$$

$$= 14 \checkmark$$

$n = 4:$

$$\underset{\text{LHS}}{\sum_{k=1}^{n} k^2} = \underbrace{1^2 + 2^2 + 3^2} + 4^2$$

$$= 14 + 16$$

$$= 30 \checkmark$$

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$$\text{RHS} = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

$$= \frac{4 \cdot 5 \cdot 9}{6}$$

$$= \frac{36 \cdot 5}{6}$$

$$= 6 \cdot 5$$

$$= 30 \checkmark$$

Proof by induction:

$$\boxed{\text{II}} \quad \sum_{K=1}^{n} K^2 = \frac{n\cdot(n+1)(2n+1)}{6}$$

To prove by induction, we follow 3 steps:

Step 1: Base case

Show the formula is true for first value of interest

✓ In this problem, we show $\boxed{\text{II}}$ is true for base case $n=1$.

Step 2: Induction hypothesis

Assume $\sum_{K=1}^{n} K^2 = \frac{n\cdot(n+1)(2n+1)}{6}$

(Assume P is true)

Step 3: Induction step

Consider case $n+1$:

$$\sum_{K=1}^{n+1} K^2 = \boxed{\underset{(K=1)}{1^2} + \underset{(K=2)}{2^2} + \cdots + \underset{(K=n)}{n^2}} + \underset{(K=n+1)}{(n+1)^2}$$

$$= \boxed{\sum_{K=1}^{n} K^2} + (n+1)^2$$

$$= \frac{n\cdot(n+1)\cdot(2n+1)}{6} + (n+1)^2$$

$$= (n+1)\left[\frac{n\cdot(2n+1)}{6} + (n+1)\right]$$

$$= (n+1)\left[\frac{2n^2+n}{6} + \frac{6n+6}{6}\right]$$

$$= (n+1)\left[\frac{2n^2 + n + 6n + 6}{6}\right]$$

$$= (n+1)\cdot\left[\frac{2n^2 + 7n + 6}{6}\right]$$

(see next page)

$$= \frac{1}{6}\cdot(n+1)(n+2)(2n+3)$$

$$\sum_{k=1}^{n+1} k^2 = \frac{1}{6}(n+1)(n+1+1)(2(n+1)+1)$$

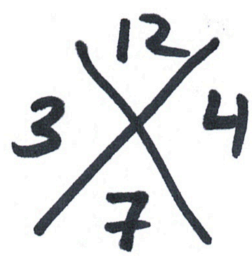
✓

Side note: Factor $\frac{2n^2 + 7n + 6}{6}$

Recall $ax^2 + bx + c$ is standard form

$\Rightarrow a = 2, \; b = 7, \; c = 6$

Let's use ac-method:


$2n^2 + 7n + 6$

$= 2n^2 + 4n + 3n + 6$

$= 2n(n+2) + 3(n+2)$

$\Rightarrow (n+2)(2n+3)$

Logical structure of induction:

Show a statement in the form

Conditional Statement

$$\boxed{P} \Rightarrow \boxed{Q}$$

"if P, then Q"

antecedent

consequent

$\boxed{P}$ is a proposition

$\boxed{Q}$ is a proposition

To "prove" a conditional statement, we think about truth tables:

| P | Q | $P \Rightarrow Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

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Proof that $P \Rightarrow Q$ is true

counter example to demonstrate $P \Rightarrow Q$ is false (conjecture does not hold true) eg. P : I am physically in California at this moment

Q : I am physically in the US at this moment.

$$P \Rightarrow Q$$

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Induction conjecture

Base case: $n=1$ is true &

If formula holds true for case $n$,

then formula holds true for case $n+1$

- If $\boxed{P}$, then $\boxed{Q}$

antecedent

consequent

- $\boxed{P} \Rightarrow \boxed{Q}$

Check this out:

- Suppose I know formula is true for the first case $n = 1$ (base)

- Suppose I know with all that I am that the induction conjecture is true: i.e. suppose

  If formula is true for case $n$, then it must be true for case $n+1$.

- To finish a proof by induction, we WTS

$$\boxed{P} \Rightarrow \boxed{Q}$$

To show inductuction conjecture is true, we :

Induction Hypothesis

- ☐ Start by assumng $\boxed{P}$ is true

  (assume formula is true for case n)

$\Downarrow$

Induction Step

- ☐ Then show that $\boxed{Q}$ must be true

  ( show n+1 must also be true)