

1. (12 points) Let $B = A \cdot X$ where

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 2 & 2 \\ 1 & -1 & -1 \\ -2 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

A. Use the entry-by-entry version of matrix-matrix multiplication to find b_{32} . Show your steps.

$$\begin{aligned} b_{32} &= \text{Entry}_{32}(B) \\ &= \text{Entry}_{32}(A \cdot X) \\ &= \text{Row}_3(A) \cdot \text{Column}_2(X) \\ &= \left[A(3,:) \right] \cdot \left[X(:,2) \right] \end{aligned} \quad \left. \begin{array}{l} \rightarrow = [0 \quad -1 \quad 2 \quad -1] \begin{bmatrix} 2 \\ -1 \\ -2 \\ 0 \end{bmatrix} \\ = 0 \cdot 2 + -1 \cdot -1 + 2 \cdot (-2) + -1 \cdot 0 \\ = 1 - 4 = \boxed{-3 = b_{32}} \end{array} \right.$$

B. Use the row-partition version of matrix-matrix multiplication to find $B(4,:)$. Show your steps.

$$\begin{aligned} B(4,:) &= \text{Row}_4(B) \\ &= \text{Row}_4(A \cdot X) \\ &= \text{Row}_4(A) \cdot X \\ &= [0 \quad -1 \quad -1 \quad 3] \left[\begin{array}{c} \vdots \\ \vdots \end{array} \right] \end{aligned} \quad \left. \begin{array}{l} \rightarrow = 0 \cdot [2 \quad 0 \quad 0 \quad 0] \\ + -1 \cdot [0 \quad 3 \quad -1 \quad -1] \\ + -1 \cdot [0 \quad -1 \quad 2 \quad -1] \\ + 3 \cdot [0 \quad -1 \quad -1 \quad 3] \\ = [0 \quad -3 \quad 1 \quad 1] + [0 \quad 1 \quad -2 \quad 1] + [0 \quad -3 \quad -3 \quad 9] \\ = [0 \quad -5 \quad -4 \quad 11] \end{array} \right.$$

C. Use the column-partition version of matrix-matrix multiplication to find $B(:,3)$. Show your steps.

$$\text{Problem 1B) } B(4,:) = \text{Row}_4(B)$$

$$= \text{Row}_4(A \cdot X)$$

$$= \text{Row}_4(A) \cdot X$$

$$= [A(4,:)]_{1 \times 4} \cdot [X]_{4 \times 3}$$

$$= [0 \ -1 \ -1 \ 3]_{1 \times 4} \begin{bmatrix} 0 & 2 & 2 \\ 1 & -1 & -1 \\ -2 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix}_{4 \times 3}$$

$$= 0 \cdot [0 \ 2 \ 2] + -1 \cdot [1 \ -1 \ -1] + -1 \cdot [-2 \ -2 \ 0] + 3 [1 \ 0 \ 3]$$

$$= [0 \ 0 \ 0] + [-1 \ 1 \ 1] + [-2 \ -2 \ 0] + [3 \ 0 \ 9]$$

$$= [0 + -1 + 2 + 3 \mid 0 + 1 + 2 + 0 \mid 0 + 1 + 0 + 9]_{1 \times 3}$$

$$= \boxed{\begin{bmatrix} 4 & 3 & 10 \end{bmatrix}} = B(4,:)$$

Check: Use calculator or compute using entry-by-entry.

$$b_{41} = A(4,:) \cdot X(:,1)$$

$$= [0 \ -1 \ -1 \ 3] \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= 0 \cdot 0 + -1 \cdot 1 + -1 \cdot -2 + 3 \cdot 1$$

$$= -1 + 2 + 3 = 4 \checkmark$$

$$b_{42} = A(4,:) \cdot X(:,2)$$

=

(2)

$$\text{Problem 1C) } B(:,3) = \text{Column}_3(B)$$

$$= \text{Column}_3(A \cdot X)$$

$$= A \cdot \text{Column}_3(X)$$

$$= [A] \cdot [X(:,3)]$$

4×4 4×1

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}$$

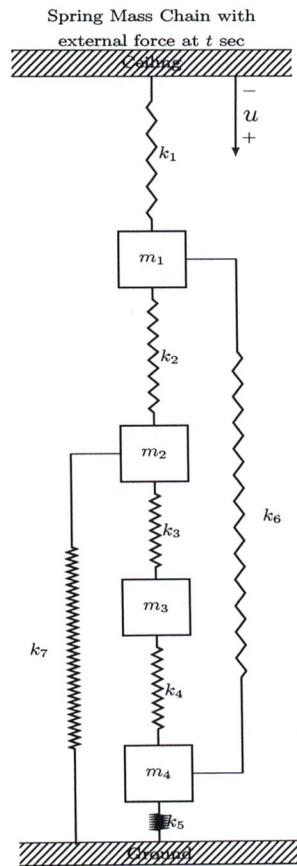
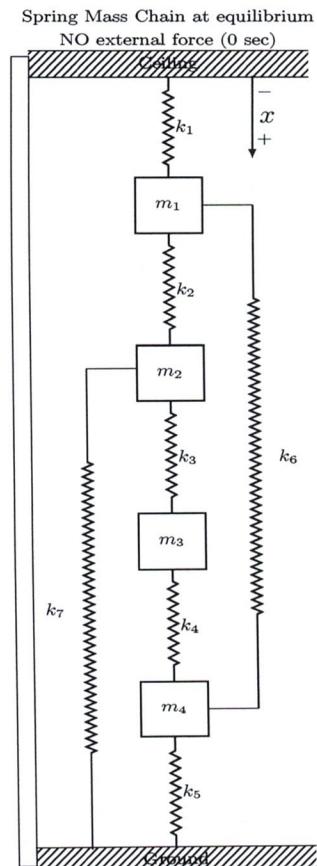
$$= 2 \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + -1 \cdot \begin{bmatrix} 0 \\ 3 \\ -1 \\ -1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ -3 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+0+0 \\ 0+-3+0+-3 \\ 0+1+0+-3 \\ 0+1+0+9 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ -6 \\ -2 \\ 10 \end{bmatrix}} = B(:,3)$$

Check: Use calculator or compute entry-by-entry.

For problems 2 - 6 below, consider the following model for a 4-mass, 7-spring chain. Note that positive positions and positive displacements are marked in the downward direction. Also assume that the mass of each spring is zero and that these springs satisfy the ideal version of Hooke's law exactly. Finally, assume that the masses move only in one axis and that the masses do not rotate in this system.



2. (3 points) Generate vector models (using appropriate matrices and vectors) to define

$$\mathbf{x}_0, \mathbf{x}(t), \text{ and } \mathbf{u}(t)$$

where these vectors represent the equilibrium position vector, the positions of each mass at time t , and the displacement vector, respectively (as discussed in class and in our lesson notes).

Let $\vec{\mathbf{x}}_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4 \times 1}$ where x_i is center of mass i at equilibrium; for $i \in [4]$.

Let $\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}_{4 \times 1}$ where $x_i(t)$ is center of mass i at time t .

$$\text{Let } \vec{\mathbf{u}}(t) = \vec{\mathbf{x}}(t) - \vec{\mathbf{x}}_0 = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - x_1 \\ x_2(t) - x_2 \\ x_3(t) - x_3 \\ x_4(t) - x_4 \end{bmatrix}$$

3. (3 points) Show how to calculate the elongation vector $\mathbf{e}(t)$ as a matrix-vector product

$$\mathbf{e}(t) = A \cdot \mathbf{u}(t)$$

Write the entry-by-entry definition of matrix A and explain how you derived the equation for each coefficient $e_i(t)$ in this vector. Your answer should include specific references to the diagrams below. As a hint, remember there should be one entry of $\mathbf{e}(t)$ for each spring in the system.

Spring 1 Elongation Diagrams	Spring 2 Elongation Diagram	Spring 3 Elongation Diagram	Spring 4 Elongation Diagram
$e_1(t) = u_1(t)$	$e_2(t) = u_2(t) - u_1(t)$	$e_3(t) = u_3(t) - u_2(t)$	$e_4(t) = u_4(t) - u_3(t)$

Spring 5 Elongation diagram	Spring 6 Elongation diagram	Spring 7 Elongation diagram
$e_5(t) = -u_4(t)$	$e_6(t) = u_4(t) - u_1(t)$	$e_7(t) = -u_2(t)$

$$\vec{e}(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \\ e_6(t) \\ e_7(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) - u_1(t) \\ u_3(t) - u_2(t) \\ u_4(t) - u_3(t) \\ -u_4(t) \\ u_4(t) - u_1(t) \\ -u_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} = A \cdot \vec{u}(t)$$

4x1
7x4

4. (3 points) Show how to calculate the spring force vector $\mathbf{f}_s(t)$ as a matrix-vector product

$$\mathbf{f}_s(t) = C \cdot \mathbf{e}(t)$$

Write the entry-by-entry definition of matrix C and discuss how Hooke's law is used to create the vector of forces for each spring.

$$\vec{\mathbf{f}}_s(t) = \begin{bmatrix} f_{s1}(t) \\ f_{s2}(t) \\ f_{s3}(t) \\ f_{s4}(t) \\ f_{s5}(t) \\ f_{s6}(t) \\ f_{s7}(t) \end{bmatrix} = \begin{bmatrix} k_1 e_1(t) \\ k_2 e_2(t) \\ k_3 e_3(t) \\ k_4 e_4(t) \\ k_5 e_5(t) \\ k_6 e_6(t) \\ k_7 e_7(t) \end{bmatrix}$$

$$= \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_7 \end{bmatrix}_{7 \times 7} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \\ e_6(t) \\ e_7(t) \end{bmatrix}_{7 \times 1}$$

$$= C \cdot \vec{\mathbf{e}}(t)$$

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5. (3 points) Create “free-body” diagrams that show all forces acting on each mass m_i . Use these diagrams to derive the vector

$$\mathbf{y}(t) = -A^T \cdot \mathbf{f}_s(t)$$

of internal forces. Also, show how to combine your equation for $\mathbf{y}(t)$ with equations from parts B and C to form the stiffness matrix K .

6. (4 points) Use Newton's second law to derive the matrix equation

$$M \cdot \ddot{\mathbf{u}}(t) + K \cdot \mathbf{u}(t) = \mathbf{f}_e(t)$$

where $\mathbf{f}_e(t)$ represents the vector of external forces on each mass. Be sure to give entry-by-entry definitions of the matrix M and the matrix K .

$$\begin{aligned} \vec{\sum F} &= \begin{bmatrix} \sum F_1 \\ \sum F_2 \\ \sum F_3 \\ \sum F_4 \end{bmatrix} = \begin{bmatrix} m_1 \ddot{u}_1(t) \\ m_2 \ddot{u}_2(t) \\ m_3 \ddot{u}_3(t) \\ m_4 \ddot{u}_4(t) \end{bmatrix} = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \begin{bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \\ \ddot{u}_4(t) \end{bmatrix} \\ &\quad \underbrace{\qquad\qquad\qquad}_{M} \qquad\qquad\qquad \underbrace{\qquad\qquad\qquad}_{\ddot{\mathbf{u}}(t)} \end{aligned}$$

We also know from problem 5 that

$$\begin{aligned} \vec{\sum F} &= \begin{bmatrix} \sum F_1 \\ \sum F_2 \\ \sum F_3 \\ \sum F_4 \end{bmatrix} = - \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_{s_1}(t) \\ f_{s_2}(t) \\ f_{s_3}(t) \\ f_{s_4}(t) \\ f_{s_5}(t) \\ f_{s_6}(t) \\ f_{s_7}(t) \end{bmatrix} + \begin{bmatrix} f_{e_1}(t) \\ f_{e_2}(t) \\ f_{e_3}(t) \\ f_{e_4}(t) \end{bmatrix} \end{aligned}$$

$$\Rightarrow \vec{\sum F} = M \cdot \ddot{\mathbf{u}}(t) = -A^T \vec{f}_s(t) + \vec{f}_e(t)$$

$$\Rightarrow M \cdot \ddot{\mathbf{u}}(t) = -A^T \cdot C \cdot \ddot{\mathbf{e}}(t) + \vec{f}_e(t)$$

$$\Rightarrow M \cdot \ddot{\mathbf{u}}(t) = -A^T \cdot C_1 \cdot A \cdot \ddot{\mathbf{u}}(t) + \vec{f}_e(t) \quad \text{let } K = A^T \cdot C_1 \cdot A$$

$$\Rightarrow M \cdot \ddot{\mathbf{u}}(t) = -K \cdot \ddot{\mathbf{u}}(t) + \vec{f}_e(t)$$

$$\Rightarrow M \cdot \ddot{\mathbf{u}}(t) + K \cdot \ddot{\mathbf{u}}(t) = \vec{f}_e(t)$$

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$$K = A^T C A$$

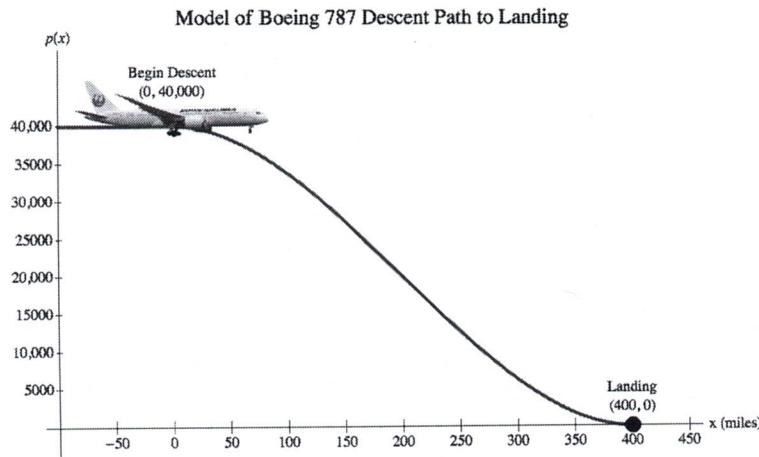
$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 & 0 & 0 & 0 \\ -k_2 & k_2 & 0 & 0 \\ 0 & -k_3 & k_3 & 0 \\ 0 & 0 & -k_4 & k_4 \\ 0 & 0 & 0 & -k_5 \\ -k_6 & 0 & 0 & k_6 \\ 0 & -k_7 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 + k_2 + k_6 & -k_2 & 0 & -k_6 \\ -k_2 & k_2 + k_3 + k_7 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 \\ -k_6 & 0 & -k_4 & k_4 + k_5 + k_6 \end{bmatrix}_{4 \times 4}$$

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7. (6 points) Suppose we are modeling the descent path for a Boeing 787 airplane landing in SFO. We can visualize this modeling problem as follows:



In this case, we choose to model the descent path by a cubic polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \quad \Leftrightarrow \quad p'(x) = a_1 + 2a_2x + 3a_3x^2$$

Here $p(x)$ represents the altitude (in feet) of the airplane after it has travelled x miles in the horizontal direction. We set $x = 0$ miles when the airplane begins its descent and notice that $x = 400$ miles at the landing point. To determine the unknown coefficients, we impose the following conditions:

Condition	Verbal Description	Equation
i.	The cruising altitude is 40000 ft at the start of the descent	$p(0) = 40000$
ii.	The tangent line to the descent path is horizontal at the start of the descent	$p'(0) = 0$
iii.	The tangent line to the descent path is horizontal at the landing point	$p'(400) = 0$
iv.	The landing point has an altitude of 0	$p(400) = 0$

Create a system of 4 equations in 4 unknowns using the conditions described above. Then, state this system as a linear-systems problem $Ax = b$ arising from your four equations. Explicitly identify matrix $A \in \mathbb{R}^{4 \times 4}$ and vector $b \in \mathbb{R}^4$.

$$\text{Equation 1: } p(0) = 40000 \Leftrightarrow 1 \cdot a_0 + 0 \cdot a_1 + 0 \cdot a_2 + 0 \cdot a_3 = 40000$$

$$\text{Equation 2: } p'(0) = 0 \Leftrightarrow 0 \cdot a_0 + 1 \cdot a_1 + 0 \cdot a_2 + 0 \cdot a_3 = 0$$

$$\text{Equation 3: } p'(400) = 0 \Leftrightarrow 0 \cdot a_0 + 1 \cdot a_1 + 400 \cdot a_2 + 480,000a_3 = 0$$

$$\text{Equation 4: } p(400) = 0 \Leftrightarrow 1 \cdot a_1 + 400 \cdot a_2 + 160,000 \cdot a_3 + 64,000,000 \cdot a_4 = 0$$

Notice, we can translate this into a linear system problem
in the form $A \cdot \tilde{x} = \tilde{b}$ with the following structure:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 400 & 480000 \\ 1 & 400 & 160000 & 64000000 \end{bmatrix}_{4 \times 4} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_{4 \times 1} = \begin{bmatrix} 40,000 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4 \times 1}$$

For problems 8, 9, and 10, choose two out of these three problems you want me to grade for your first attempt during our in-class exam session. For the problem you would like to skip grading for your in-class attempt, please mark a big "X" through that problem. For the problem you skip, you can submit your solutions in your exam corrections. For now, focus on the two of these problems that you feel most comfortable with and give your best effort.

8. (8 points) Consider the following nonsingular linear-systems problem

$$\underbrace{\begin{bmatrix} 3 & 1 & -2 \\ -3 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}}_b$$

Transform this system into an equivalent system $U\mathbf{x} = \mathbf{y}$ using left-multiplication by elementary matrices, where $U \in \mathbb{R}^{3 \times 3}$ is upper-triangular with nonzero diagonal elements. Then, solve this equivalent system using backward substitution.

Let's start by rewriting the problem in my own handwriting

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 3 & 1 & -2 \\ -3 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix}_{3 \times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}_{3 \times 1}$$

pivot 1

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ 0 & 2 & -2 \\ -6 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ 0 & 2 & -2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 10 \end{bmatrix}$$

\Rightarrow

$$\underbrace{\begin{bmatrix} 3 & 1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & -1 \end{bmatrix}}_{U} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}}_{\vec{g}}$$

Backward substitution: Initialization

$$-1 \cdot x_3 = 2 \Rightarrow \boxed{x_3 = -2}$$

$$\text{Step 1: } 2 \cdot x_2 + -2 \cdot x_3 = 8$$

$$\Rightarrow 2 \cdot x_2 = 8 - (-2 \cdot x_3)$$

$$\Rightarrow x_2 = \frac{8 - (-2) \cdot (-2)}{2}$$

$$\Rightarrow x_2 = \frac{8 - 4}{2} = \frac{4}{2}$$

$$\Rightarrow \boxed{x_2 = 2}$$

$$\text{Step 2: } 3 \cdot x_1 + 1 \cdot x_2 + -2 \cdot x_3 = 3$$

$$\Rightarrow 3 \cdot x_1 = 3 - 1 \cdot x_2 - (-2) \cdot x_3$$

$$\Rightarrow x_1 = \frac{3 - 1 \cdot (2) - (-2) \cdot (-2)}{3}$$

$$\Rightarrow x_1 = \frac{3 - 2 - 4}{3}$$

$$\Rightarrow x_1 = \frac{-3}{3}$$

$$\Rightarrow x_1 = -1$$

$$\Rightarrow \vec{x}^L = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$\text{Check: } A \cdot \vec{x} = \begin{bmatrix} 3 & 1 & -2 \\ -3 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}$$

$$= -1 \cdot \begin{bmatrix} 3 \\ -3 \\ -6 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + -2 \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} = \vec{b} \checkmark$$

9. (8 points) Find the LU Factorization of the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}.$$

Show your work and explain your steps.

Let's transform A to uppertriangular form using Gauss transforms.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix}}_{L_1} \cdot \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 15/4 & 3/4 \\ 0 & 3/4 & 15/4 \end{bmatrix}$$

$L_1 \cdot A$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/5 & 1 \end{bmatrix}}_{L_2} \cdot \underbrace{\begin{bmatrix} 1 & 4 & 1 \\ 0 & 15/4 & 3/4 \\ 0 & 3/4 & 15/4 \end{bmatrix}}_{L_1 \cdot A} = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 15/4 & 3/4 \\ 0 & 0 & 18/5 \end{bmatrix}$$

$L_2 \cdot L_1 \cdot A$

U

$$\text{Note 1: } -\frac{3}{4} \div \frac{15}{4} = -\frac{3}{4} \cdot \frac{4}{15} = -\frac{3}{15}$$

$$\Rightarrow -l_{32} = -\frac{3}{15} = -\frac{1}{5}$$

$$\Rightarrow l_{32} = \frac{1}{5}$$

$$\text{Note 2: } -\frac{1}{5} \cdot \frac{3}{4} + \frac{15}{4} = -\frac{3}{20} + \frac{15}{4}$$

$$= \frac{-3 + 75}{20}$$

$$= \frac{72}{20} = \frac{18}{5}$$

Now we have

$$L_2 \cdot L_1 \cdot A = UT$$

But, by problem 10, we know L_1 and L_2 are invertible

and we have

$$A = L_1^{-1} \cdot L_2^{-1} \cdot UT = L \cdot UT$$

where $L = L_1^{-1} \cdot L_2^{-1}$. Moreover

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/4 & 0 & 1 \end{bmatrix} \quad \text{and} \quad L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/5 & 1 \end{bmatrix}$$

We also have that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/4 & 1/5 & 1 \end{bmatrix}$$

Then, the LUT factorization of A is given as

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/4 & 1/5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 0 & 15/4 & 3/4 \\ 0 & 0 & 18/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1.00 & 0 & 0 \\ 0.25 & 1.00 & 0 \\ 0.25 & 0.20 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 0 & 3.75 & 0.75 \\ 0 & 0 & 3.60 \end{bmatrix}$$

10. (8 points) Let $n \in \mathbb{N}$ with $n \geq 6$. Let $k \in \mathbb{N}$ where $k \in \{1, 2, \dots, (n-1)\}$. Suppose we have a set of scalars $\{\ell_{k+1}, \ell_{k+2}, \dots, \ell_n\} \subset \mathbb{R}$. Let's define the vector $\tau_k \in \mathbb{R}^n$ with

$$\tau_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \ell_{k+1} \\ \vdots \\ \ell_n \end{bmatrix} \quad \text{where} \quad \tau_k(j, 1) = \begin{cases} 0 & \text{if } 1 \leq j \leq k, \\ \ell_j & \text{if } k+1 \leq j \leq n \end{cases}$$

Using this definition, notice that the first k entries of τ_k are zero. Let's define the matrix

$$L_k = I_n - \tau_k \mathbf{e}_k^T \quad \text{where} \quad \mathbf{e}_k = I_n(:, k).$$

Describe the structure of the matrix L_k by identifying each of the following:

- How many nonzero entries can be found in matrix L_k ?
- Exactly where are these nonzero entries, matrix L_k ?
- What are the values of each nonzero entry in matrix L_k ?

Then, show that

$$L_k^{-1} = I_n + \tau_k \mathbf{e}_k^T.$$

$$L_k = I_n - \tau_k \cdot \tilde{\mathbf{e}}_k^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Row K = $\boxed{0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ 0 \ \cdots \ 0}$

Column K

Entry-by-entry def of L_k

Let $i, j \in [n]$.

Case 1: Assume $j \neq k$

$$\text{Entry}_{ij}(L_k) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Case 2: Assume $j = k$

$$\text{Entry}_{ij}(L_k) = \begin{cases} 0 & \text{if } i < k \\ 1 & \text{if } i = k \\ -\ell_i & \text{if } i > k \end{cases}$$

$$\text{Let } M_K = I_n + \tilde{\tau}_K \cdot \tilde{e}_K^\top$$

$$L_K \cdot M_K = (I_n - \tilde{\tau}_K \cdot \tilde{e}_K^\top) \cdot (I_n + \tilde{\tau}_K \cdot \tilde{e}_K^\top)$$

$$= I_n \cdot (I_n + \tilde{\tau}_K \cdot \tilde{e}_K^\top) - \tilde{\tau}_K \cdot \tilde{e}_K^\top (I_n + \tilde{\tau}_K \cdot \tilde{e}_K^\top)$$

$$= I_n + \tilde{\tau}_K \cdot \tilde{e}_K^\top - \tilde{\tau}_K \cdot \tilde{e}_K^\top - \tilde{\tau}_K (\tilde{e}_K^\top / \tilde{\tau}_K) \tilde{e}_K^\top$$

$$= I_n$$

Side note:

$$\tilde{e}_K^\top \cdot \tilde{\tau}_K = [0 \cdots 0 \boxed{1} 0 \cdots 0]_{1 \times n}$$

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \boxed{0} \\ \vdots \\ 0_n \end{bmatrix}_{n \times 1} \cdot \begin{bmatrix} \text{entry } k \\ \vdots \\ \text{entry } k+1 \\ \vdots \\ \text{entry } n \end{bmatrix}_{1 \times n}$$

$$\Rightarrow \tilde{e}_K^\top \cdot \tilde{\tau}_K = \sum_{j=1}^n \text{Entry}_{ij}(\tilde{e}_K^\top) \cdot \text{Entry}_{ji}(\tilde{\tau}_K)$$

$$= \sum_{j=1}^{k-1} \text{Entry}_{ij}(\tilde{e}_K^\top) \cdot \text{Entry}_{ji}(\tilde{\tau}_K) + 1 \cdot 0$$

$$+ \sum_{j=k+1}^n \text{Entry}_{ij}(\tilde{e}_K^\top) \cdot \text{Entry}_{ji}(\tilde{\tau}_K)$$

$$= \sum_{j=1}^k 0 \cdot 0 + 1 \cdot 0 + \sum_{j=k+1}^n 0 \cdot l_j$$

$$= 0$$

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But now we have

$$L_K \cdot M_K = I_n \Rightarrow \tilde{L}_K^{-1} = M_K$$

$$\Rightarrow \tilde{L}_K^{-1} = I_n + \tilde{\tau}_K \cdot \tilde{e}_K^\top \text{ as}$$

was claimed in the first place!!

Challenge Problem

11. (Optional, Extra Credit, Challenge Problem)

Let $n \in \mathbb{N}$ and $K \in \mathbb{R}^{n \times n}$ with $K^T = K$. Suppose that $\mathbf{x}^T \cdot K \cdot \mathbf{x} > 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$. Define the quadratic form $q : \mathbb{R}^n \rightarrow \mathbb{R}$ using the matrix equation

$$q(\mathbf{x}) = \mathbf{x}^T \cdot K \cdot \mathbf{x} - 2\mathbf{x}^T \mathbf{f} + c$$

for constant $c \in \mathbb{R}$ and constant vector $\mathbf{f} \in \mathbb{R}^n$.

- A. Show that K is invertible.
- B. Show that $q(\mathbf{x})$ has a unique minimizer $\mathbf{x}^* \in \mathbb{R}^n$ such that $q(\mathbf{x}^*) < q(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.