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# Math 2B: Applied Linear Algebra Fall 2019 Exam 2: Tuesday 11/19/2019 

## How long is this exam?

- This exam is scheduled for a 135 minute period.
- Make sure you have 6 sheets of paper (12 pages front and back) including this cover page.
- There are a total of 10 separate questions on this exam including:
- 6 Free-response questions (50 points)
- 1 Optional, extra credit challenge problem (5 points)


## How will your written work be graded on these questions?

- Your work should show evidence of original thought and deep understanding. Work that too closely resembles the ideas presented in Jeff's lesson notes will likely NOT earn top scores. Work that does not demonstrate individualized, nuanced understanding will likely NOT earn top scores.
- Read the directions carefully. Your work will be graded based on what you are being asked to do.
- In order to earn a top score, please show all your work. In most cases, a correct answer with no supporting work will NOT earn top scores. What you write down and how you write it are the most important means of getting a good score on this exam.
- Neatness and organization are IMPORTANT! Do your best to make your work easy to read.
- You will be graded on accurate use of the notation we studied in this class.


## What can you use on this exam?

- You may use no more than six note sheets (double-sided) or twelve note sheets (single-sided).
- Each note sheet is to be no larger then 11 -inches by 8.5 -inches (standard U.S. letter-sized paper).
- You must be the author of your own notes. and your note sheets must be handwritten (in YOUR OWN handwriting).
- PLEASE SUBMIT ALL OF YOUR NOTE SHEETS WITH YOUR EXAM.
- You are allowed to use calculators for this exam. Examples of acceptable calculators include TI 83, TI 84, and TI 86 calculators. You are not allowed to use any calculator with a Computer Algebra System including TI 89 and TI NSpire. If you have a question, please ask your instructor about this.


## What other rules govern your participation during this exam?

- PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO!
- It is a violation of the Foothill Academic Integrity Code to, in any way, assist another person in the completion of this exam. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted. Thank you for your cooperation.
- No notes (other than your note sheets), books, or classmates can be used as resources for this exam.
- Please turn off your cell phones during this exam. No cell phones will be allowed on your desk.

1. (12 points) Let $B=A \cdot X$ where

$$
A=\left[\begin{array}{rrrr}
2 & 0 & 0 & 0 \\
0 & 3 & -1 & -1 \\
0 & -1 & 2 & -1 \\
0 & -1 & -1 & 3
\end{array}\right]
$$

$$
X=\left[\begin{array}{rrr}
0 & 2 & 2 \\
1 & -1 & -1 \\
-2 & -2 & 0 \\
1 & 0 & 3
\end{array}\right]
$$

A. Use the entry-by-entry version of matrix-matrix multiplication to find $b_{32}$. Show your steps.
B. Use the row-partition version of matrix-matrix multiplication to find $B(4,:)$. Show your steps.
C. Use the column-partition version of matrix-matrix multiplication to find $B(:, 3)$. Show your steps.

For problems 2-6 below, consider the following model for a 4 -mass, 7 -spring chain. Note that positive positions and positive displacements are marked in the downward direction. Also assume that the mass of each spring is zero and that these springs satisfy the ideal version of Hooke's law exactly. Finally, assume that the massed move only in one axis and that the masses do not rotate in this system.

2. (3 points) Generate vector models (using appropriate matrices and vectors) to define

$$
\mathbf{x}_{0}, \mathbf{x}(t), \text { and } \mathbf{u}(t)
$$

where these vectors represent the equilibrium position vector, the positions of each mass at time $t$, and the displacement vector, respectively (as discussed in class and in our lesson notes).
3. (3 points) Show how to calculate the elongation vector $\mathbf{e}(t)$ as a matrix-vector product

$$
\mathbf{e}(t)=A \cdot \mathbf{u}(t)
$$

Write the entry-by-entry definition of matrix $A$ and explain how you derived the equation for each coefficient $e_{i}(t)$ in this vector. Your answer should include specific references to the diagrams below. As a hint, remember there should be one entry of $\mathbf{e}(t)$ for each spring in the system.
4. (3 points) Show how to calculate the spring force vector $\mathbf{f}_{s}(t)$ as a matrix-vector product

$$
\mathbf{f}_{s}(t)=C \cdot \mathbf{e}(t)
$$

Write the entry-by-entry definition of matrix $C$ and discuss how Hooke's law is used to create the vector of forces for each spring.
5. (3 points) Create "free-body" diagrams that show all forces acting on each mass $m_{i}$. Use these diagrams to derive the vector

$$
\mathbf{y}(t)=-A^{T} \cdot \mathbf{f}_{s}(t)
$$

of internal forces. Also, show how to combine your equation for $\mathbf{y}(t)$ with equations from parts B and C to form the stiffness matrix $K$.
6. (4 points) Use Newton's second law to derive the matrix equation

$$
M \cdot \ddot{\mathbf{u}}(t)+K \cdot \mathbf{u}(t)=\mathbf{f}_{e}(t)
$$

where $\mathbf{f}_{e}(t)$ represents the vector of external forces on each mass. Be sure to give entry-by-entry definitions of the matrix $M$ and the matrix $K$.
7. (6 points) Suppose we are modeling the descent path for a Boeing 787 airplane landing in SFO. We can visualize this modeling problem as follows:


In this case, we choose to model the descent path by a cubic polynomial

$$
p(x)=a_{0}+a_{1} x+a_{2} x^{2}+a_{3} x^{3}
$$

Here $p(x)$ represents the altitude (in feet) of the airplane after it has travelled $x$ miles in the horizontal direction. We set $x=0$ miles when the airplane begins its descent and notice that $x=400$ miles at the landing point. To determine the unknown coefficients, we impose the following conditions:

| Condition | Verbal Description | Equation |
| :---: | :--- | :---: |
| i. | The cruising altitude is 40000 ft at the start of the descent | $p(0)=40000$ |
| ii. | The tangent line to the descent path is horizontal at the start of the descent | $p^{\prime}(0)=0$ |
| iii. | The tangent line to the descent path is horizontal at the landing point | $p^{\prime}(400)=0$. |
| iv. | The landing point has an altitude of 0 | $p(400)=0$ |

Create a system of 4 equations in 4 unknowns using the conditions described above. Then, state this system as a linear-systems problem $A \mathbf{x}=\mathbf{b}$ arising from your four equations. Explicitly identify matrix $A \in \mathbb{R}^{4 \times 4}$ and vector $\mathbf{b} \in \mathbb{R}^{4}$.

For problems 8, 9, and 10, choose two out of these three problems you want me to grade for your first attempt during our in-class exam session. For the problem you would like to skip grading for your inclass attempt, please mark a big "X" through that problem. For the problem you skip, you can submit your solutions in your exam corrections. For now, focus on the two of these problems that you feel most comfortable with and give your best effort.
8. (8 points) Consider the following nonsingular linear-systems problem

$$
\underbrace{\left[\begin{array}{rrr}
3 & 1 & -2 \\
-3 & 1 & 0 \\
-6 & 0 & 1
\end{array}\right]}_{A} \underbrace{\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]}_{\mathbf{x}}=\underbrace{\left[\begin{array}{l}
3 \\
5 \\
4
\end{array}\right]}_{\mathbf{b}}
$$

Transform this system into an equivalent system $U \mathbf{x}=\mathbf{y}$ using left-multiplication by elementary matrices, where $U \in \mathbb{R}^{3 \times 3}$ is upper-triangular with nonzero diagonal elements. Then, solve this equivalent system using backward substitution.
9. (8 points) Find the LU Factorization of the matrix

$$
A=\left[\begin{array}{lll}
4 & 1 & 1 \\
1 & 4 & 1 \\
1 & 1 & 4
\end{array}\right]
$$

Show your work and explain your steps.
10. ( 8 points) Let $n \in \mathbb{N}$ with $n \geq 6$. Let $k \in \mathbb{N}$ where $k \in\{1,2, \ldots,(n-1)\}$. Suppose we have a set of scalars $\left\{\ell_{k+1}, \ell_{k+2}, \ldots, \ell_{n}\right\} \subset \mathbb{R}$. Let's define the vector $\boldsymbol{\tau}_{k} \in \mathbb{R}^{n}$ with

$$
\boldsymbol{\tau}_{k}=\left[\begin{array}{c}
0 \\
\vdots \\
0 \\
\ell_{k+1} \\
\vdots \\
\ell_{n}
\end{array}\right] \quad \text { where }
$$

$$
\boldsymbol{\tau}_{k}(j, 1)=\left\{\begin{aligned}
0 & \text { if } 1 \leq j \leq k \\
\ell_{j} & \text { if } k+1 \leq j \leq n
\end{aligned}\right.
$$

Using this definition, notice that the first $k$ entries of $\boldsymbol{\tau}_{k}$ are zero. Let's define the matrix

$$
L_{k}=I_{n}-\boldsymbol{\tau}_{k} \mathbf{e}_{k}^{T} \quad \text { where } \quad \mathbf{e}_{k}=I_{n}(:, k)
$$

Describe the structure of the matrix $L_{k}$ by identifying each of the following:
i. How many nonzero entries can be found in matrix $L_{k}$ ?
ii. Exactly where are these nonzero entries, matrix $L_{k}$ ?
iii. What are the values of each nonzero entry in matrix $L_{k}$ ?

Then, show that

$$
L_{k}^{-1}=I_{n}+\boldsymbol{\tau}_{k} \mathbf{e}_{k}^{T} .
$$

## Challenge Problem

11. (Optional, Extra Credit, Challenge Problem)

Let $n \in \mathbb{N}$ and $K \in \mathbb{R}^{n \times n}$ with $K^{T}=K$. Suppose that $\mathbf{x}^{T} \cdot K \cdot \mathbf{x}>0$ for all nonzero $\mathbf{x} \in \mathbb{R}^{n}$. Define the quadratic form $q: \mathbb{R}^{n} \rightarrow \mathbb{R}$ using the matrix equation

$$
q(\mathbf{x})=\mathbf{x}^{T} \cdot K \cdot \mathbf{x}-2 \mathbf{x}^{T} \mathbf{f}+c
$$

for constant $c \in \mathbb{R}$ and constant vector $\mathbf{f} \in \mathbb{R}^{n}$.
A. Show that $K$ is invertible.
B. Show that $q(\mathbf{x})$ has a unique minimizer $\mathbf{x}^{\star} \in \mathbb{R}^{n}$ such that $q\left(\mathbf{x}^{\star}\right)<q(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^{n}$.

