The Eigenvalue Problem

Make the Eigenvalue Problem Resonate with Our Students

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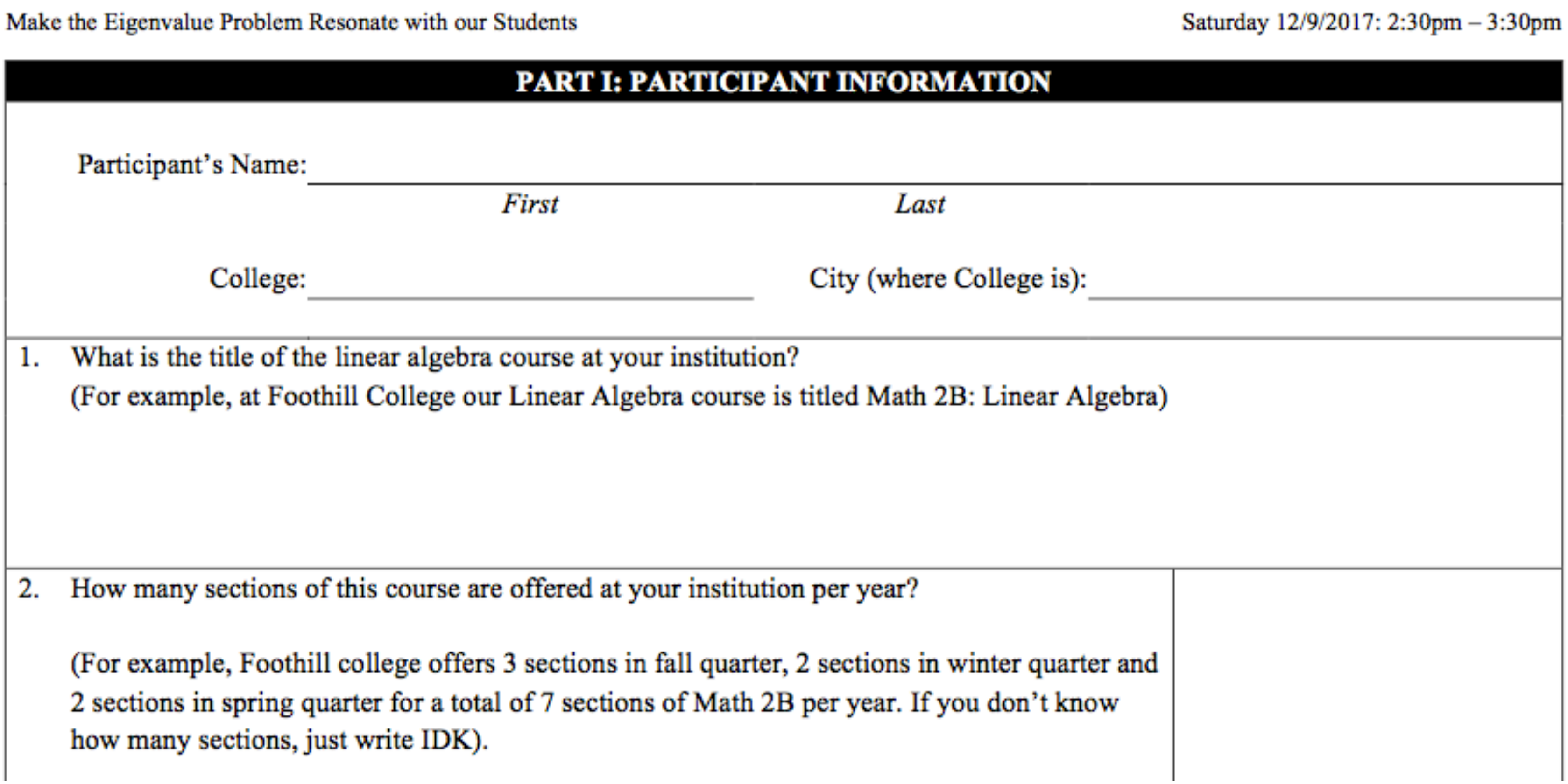
Michael McCusker, PhD
Foothill College
December 9, 2017

Getting to Know You

Please work on front of survey



Make the Eigenvalue Problem Resonate with our Students — Saturday 12/9/2017: 2:30pm – 3:30pm

PART I: PARTICIPANT INFORMATION

Participant's Name: ______________________________

First *Last*

College: ____________________ City (where College is): ____________________

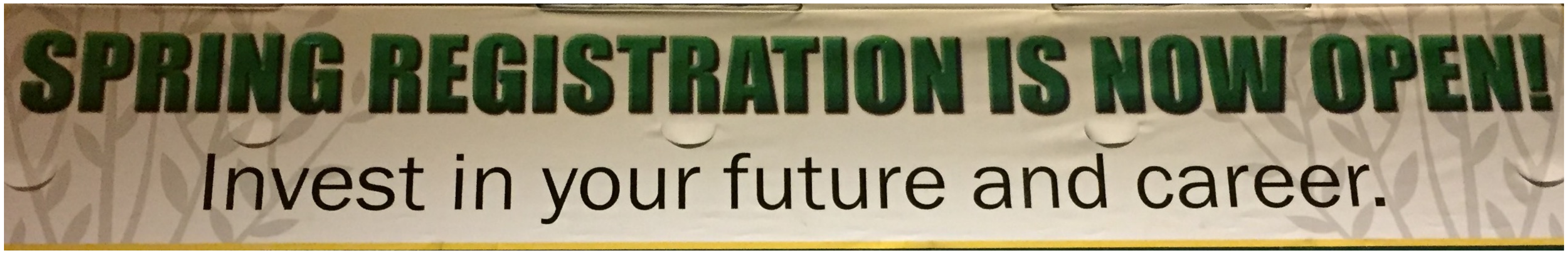
1. What is the title of the linear algebra course at your institution?
(For example, at Foothill College our Linear Algebra course is titled Math 2B: Linear Algebra)

2. How many sections of this course are offered at your institution per year?

(For example, Foothill college offers 3 sections in fall quarter, 2 sections in winter quarter and 2 sections in spring quarter for a total of 7 sections of Math 2B per year. If you don't know how many sections, just write IDK).

The Promise(s) of Math Education

What promises does the US college education system make to our students (and their families)?

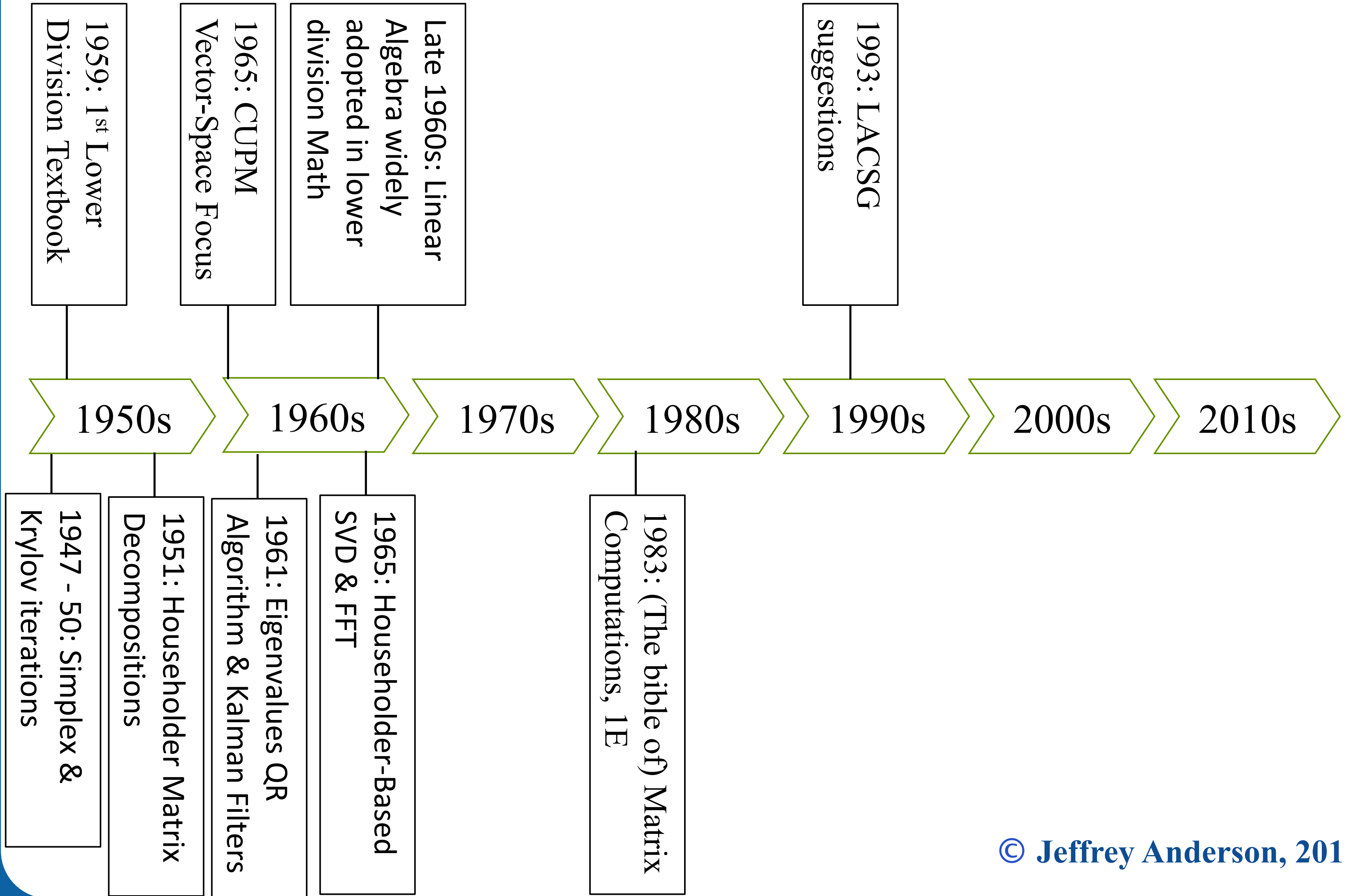


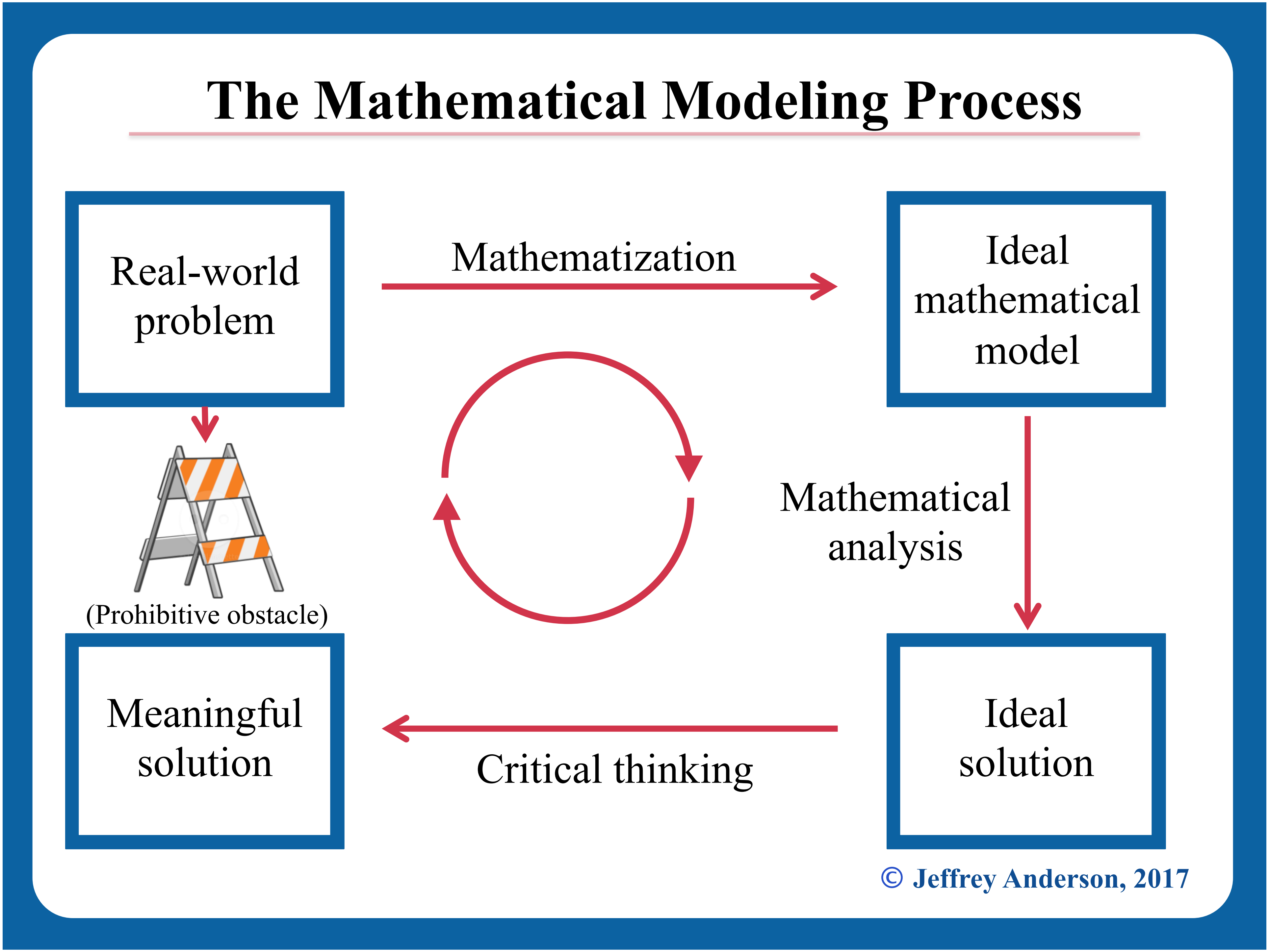
What is the purpose of math education?

Linear Algebra in Lower Division

DECLARED MAJOR	#
Business	2
Cognitive Science	1
Computer Science	30
Economics	3
Engineering (Total)	33
Bio	1
BioMedical	1
Civil	5
Electrical	6
Environmental	2
Material Science	1
Mechanical	15
Unspecified	2
English Literature	1
Math	6
Math (Applied)	2
Physics	2
Statistics	4
TOTAL	84

Themes of Linear Algebra Education




The Mathematical Modeling Process
Real-world problem
Mathematatization
Ideal mathematical model
Mathematical analysis
Ideal solution
Critical thinking
Meaningful solution
(Prohibitive obstacle)
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Six Major Problems

CALCLUS

1. $F(X) \in C^{(1)}(\mathbb{R})$

2. $\frac{d}{dx}\left[F(x)\right] = f(x)$

3. $\frac{d}{dx}\left[F(x)\right] = f(x)$

4. $\nabla\left[F(\mathbf{x})\right] = \mathbf{f}(\mathbf{x})$

5. $\nabla\left[F(\mathbf{x})\right] = \mathbf{f}(\mathbf{x})$

6. $F(x) = \sum_{n=0}^{\infty} c_n \, (x-a)^n$

LINEAR ALGEBRA

$A \in \mathbb{R}^{m \times n}$

$A\mathbf{x} = \mathbf{b}$

$A\mathbf{x} = \mathbf{b}$

$\min_{\mathbf{x} \in \mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2$

$A\mathbf{x} = \lambda\mathbf{x}$

$A = U\Sigma V^*$

Linear Algebra Calendar Re-Envisioned

$$A \in \mathbb{R}^{m\times n}$$

$$A\mathbf{x} = \mathbf{b}$$

$$A\mathbf{x} = \mathbf{b}$$

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2$$

$$A\mathbf{x} = \lambda\mathbf{x}$$

Linear Algebra Tentative Calendar

MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY
Class 1 Lesson 1: Six Major Problems		Class 2 Lesson 2: Intro to Set Theory		Class 3 Lesson 3: Relations and Functions
Class 4 Lesson 4: Vector Modeling		Class 5 Lesson 5: Vector Arithmetic		Class 6 Lesson 6: Inner Products
Class 7 Lesson 7: Vector norms		Class 8 Lesson 8: Linear Independence		Class 9 Lesson 9: Matrix Modeling
Class 10 Lesson 10: Outer Products & Elementary Matrices		Class 11 Lesson 11: Matrix Arithmetic		Class 12 Lesson 12: Matrix-Vector Multiplication
Class 13 **IN-CLASS EXAM 1** **LESSONS 0 - 11**		Class 14 Lesson 13: Four Versions of Matrix- Matrix Multiplication		Class 15 Lesson 14: Nonsingular Linear-Systems Problem
Class 16 Lesson 15: Matrix Inverses		Class 17 Lesson 16: Invertible Matrix Theorem		Class 18 Lesson 17: LU Factorization
Class 19 Lesson 18: General Linear-Systems Problem & Row Echelon Form		Class 20 Lesson 19: Solution Sets to General Linear Systems Problem		Class 21 Lesson 20: Determinants
Class 22 Lesson 21: Vector Spaces		Class 23 Lesson 22: Null & Column Space		Class 24 Lesson 23: Dimension and Rank
Class 25 Lesson 24: Least Squares		Class 26 Lesson 25: Orthogonal Sets		Class 27 Lesson 26: Orthogonal Projections
Class 28 **IN-CLASS EXAM 2** **LESSONS 12 – 25**		Class 29 Lesson 27: Gram-Schmidt Process		Class 30 Lesson 28: QR Factorization
Class 31 Lesson 29: Intro Eigenvalues		Class 32 Lesson 30: Characteristic Eq.		Class 33 Lesson 31: Diagonalization
*****FINAL EXAM** 8:00AM – 10AM***	****Finals Week****	****Finals Week****	****Finals Week****	****Finals Week****

Linear Algebra Examples Re-Envisioned

Linear Algebra Applications List

LESSONS 3: RELATIONS AND FUNCTIONS

- ☐ Convert natural numbers into binary representations
- ☐ Convert binary numbers into natural numbers
- ☐ Apply the gray scale function for storing shades of gray in a computer

LESSONS 4: VECTORS AND MODELING

- ☐ Use column vectors to create a vertex model of points in $\mathbb{R}^2$
- ☐ Use column vectors to describe data from Hooke's law experiment
- ☐ Use column vectors to capture position data for mass-spring chain
- ☐ Use column vectors to model Ohm's law experiment
- ☐ Apply Ohm's Law to describe relations between voltage and current in circuits

LESSONS 5: VECTORS ARITHMETIC

- ☐ Use scalar-vector multiplication and vector addition to model for Hooke's law
- ☐ Use vector-vector addition to create the displacement vector for a mass-spring system with n masses and $(n+1)$ springs where n = 2, 3, 4, 5

LESSONS 6: INNER PRODUCTS

- ☐ Use inner products to calculate the voltage across a circuit element given the node voltage potentials on either side of that circuit element
- ☐ Use inner products to write Kirchoff's current law for any node of an ideal circuit

LESSONS 7: MATRIX MODELING

- ☐ Create the incidence matrix for a given undirected graph
- ☐ Create incidence matrix for a given directed graph
- ☐ Identify and use the 2D wireframe model for given polygon
- ☐ Set up matrix model for a given mass-spring chain with n masses and $(n+1)$ springs
- ☐ Properly identify and apply matrix model for digital image

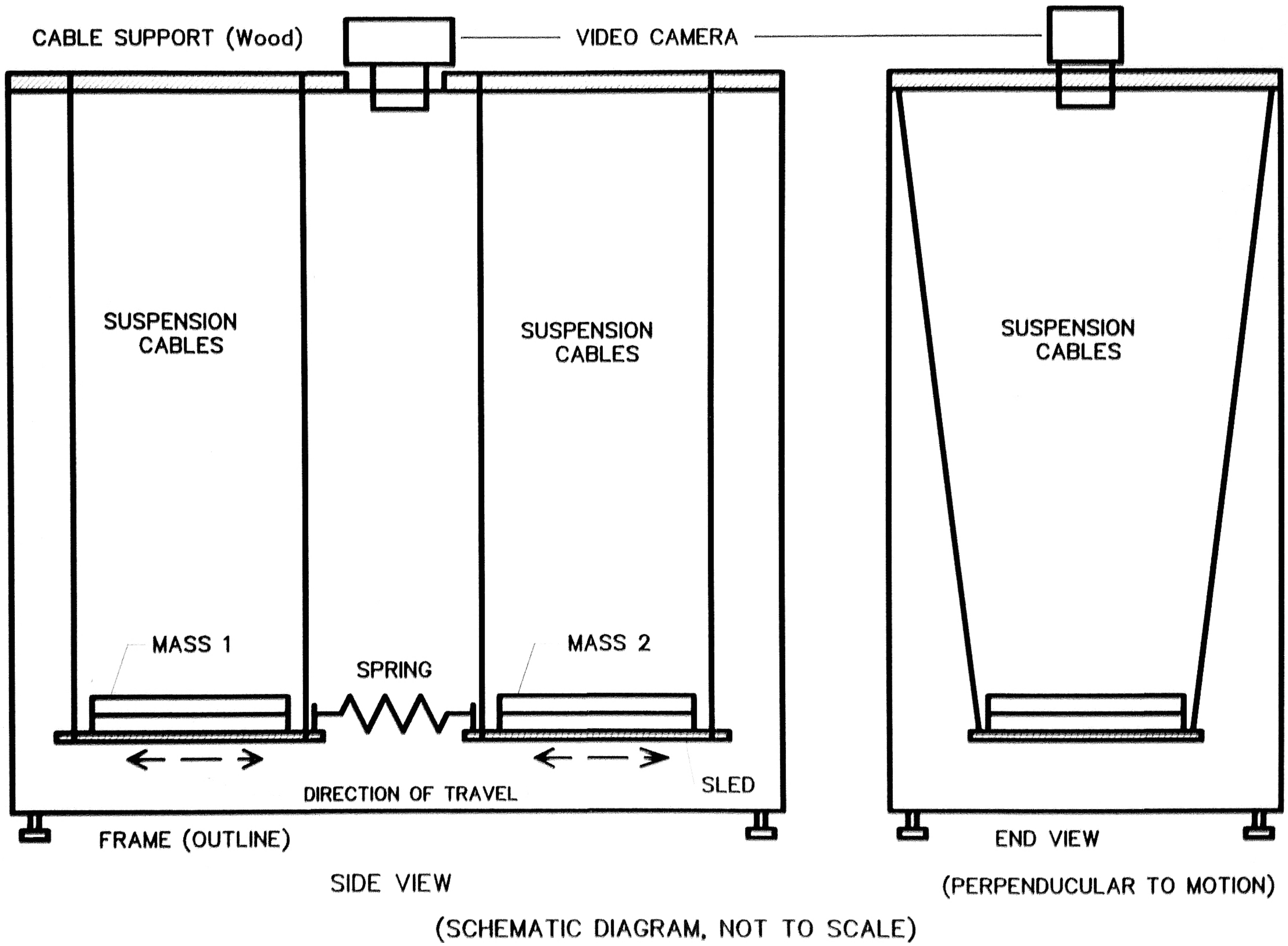
LESSON 12: MATRIX-VECTOR MULTIPLICATION

- ☐ Use matrix-vector multiplication to analyze mass-spring chains
- ☐ Use matrix-vector multiplication to calculate voltage drops across ideal circuit elements
- ☐ Use matrix-vector multiplication to state KCL at all nodes of a circuit
- ☐ Use matrix-vector multiplication to state Ohm's Law for all resistors in a circuit

LESSON 12: NONSINGULAR LINEAR-SYSTEMS PROBLEM

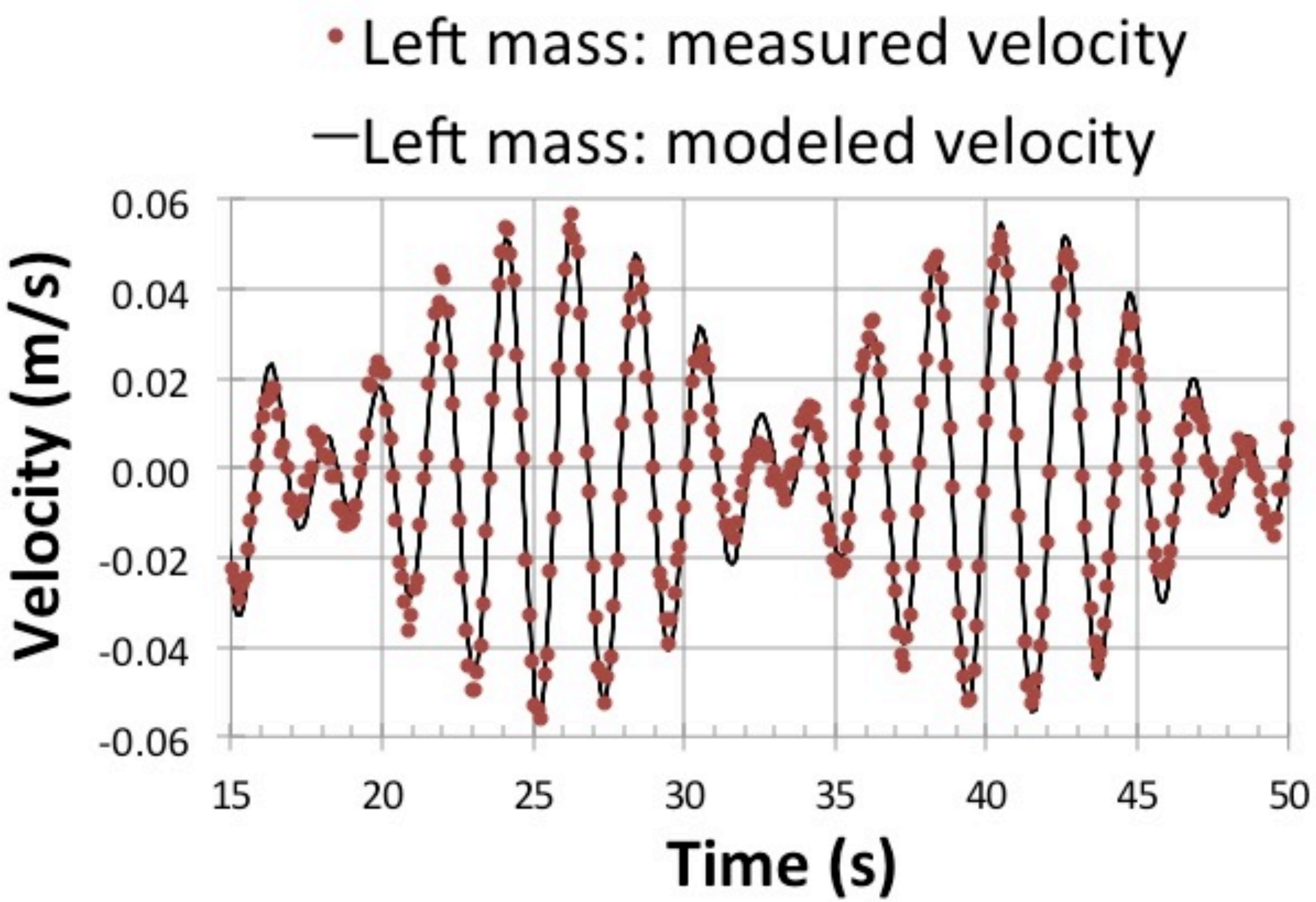
- ☐ Set up and solve a nonsingular linear-systems problem for a given mass-spring chain with n masses and $(n+1)$ springs where $n = 2, 3, 4, 5, 6$
- ☐ Set up a linear systems problems using a Vandermonde matrix for polynomial modeling.

Eigenvalue Modeling



Data Collection

Model Validation



Student Work

a. (4 pts) Generate vector models (using appropriate matrices and vectors) to define each of the following:

$$\mathbf{u}, \mathbf{e}, \mathbf{F}_s, \mathbf{y},$$

where these vectors represent the displacement vector, elongation vector, spring-force vector and net internal force vector respectively (as discussed in class).

$$u_i = x_i(t) - x_i(0) \iff \vec{u}(t) = \begin{bmatrix} x_1(t) - x_1(0) \\ x_2(t) - x_2(0) \end{bmatrix}$$

$$e_i = u_{i+1} - u_{i-1} \iff \vec{e}(t) = \begin{bmatrix} u_1(t) + 0u_2(t) \\ -u_1(t) + u_2(t) \\ 0u_1(t) - u_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \vec{u}(t) = A\vec{u}(t)$$

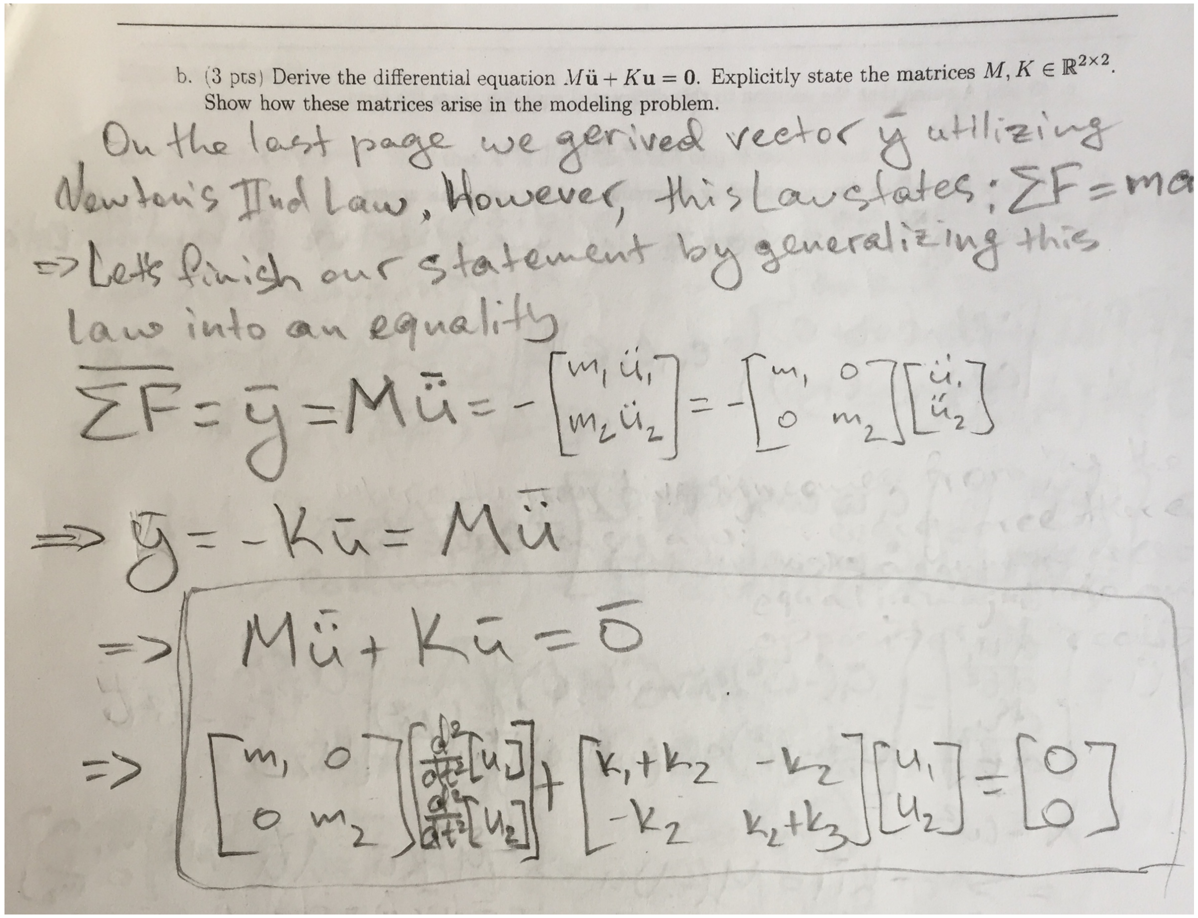
$$F_{s_i} = k_i e_i \iff \vec{F}_s = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \vec{e} = C\vec{e}$$

$$y_i = F_{s_{i+1}} - F_{s_i} \iff \vec{y} = \begin{bmatrix} -F_{s_1} + F_{s_2} + 0F_{s_3} \\ 0F_{s_1} - F_{s_2} + F_{s_3} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} F_{s_1} \\ F_{s_2} \\ F_{s_3} \end{bmatrix}$$

$$= -A^T C \vec{e}$$

$$= -A^T C A \vec{u}(t)$$

Student Work



b. (3 pts) Derive the differential equation $M\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{0}$. Explicitly state the matrices $M, K \in \mathbb{R}^{2\times 2}$. Show how these matrices arise in the modeling problem.

On the last page we gerived vector $\bar{y}$ utilizing Newton's IInd Law, However, this Law states: $\Sigma F = ma$

=> Let's finish our statement by generalizing this law into an equality

$$\overline{\Sigma F} = \bar{y} = M\bar{\ddot{u}} = -\begin{bmatrix} m_1 \ddot{u}_1 \\ m_2 \ddot{u}_2 \end{bmatrix} = -\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix}$$

$$\Rightarrow \bar{y} = -K\bar{u} = M\bar{\ddot{u}}$$

$$\Rightarrow M\bar{\ddot{u}} + K\bar{u} = \bar{0}$$

$$\Rightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \frac{d^2}{dt^2}[u_1] \\ \frac{d^2}{dt^2}[u_2] \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Student Work

a. (4 pts) Generate vector models (using appropriate matrices and vectors) to define each of the following:

$$\mathbf{u}, \mathbf{e}, \mathbf{F}_s, \mathbf{y},$$

where these vectors represent the displacement vector, elongation vector, spring-force vector and net internal force vector respectively (as discussed in class).

$$u_i = x_i(t) - x_i(0) \iff \vec{u}(t) = \boxed{\begin{bmatrix} x_1(t) - x_1(0) \\ x_2(t) - x_2(0) \end{bmatrix}}$$

$$e_i = u_i - u_{i-1} \iff \vec{e}(t) = \begin{bmatrix} u_1(t) + 0u_2(t) \\ -u_1(t) + u_2(t) \\ 0u_1(t) - u_2(t) \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \vec{u}(t)} = A\vec{u}(t)$$

$$F_{s_i} = k_i e_i \iff \vec{F}_s = \boxed{\begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \vec{e}} = C\vec{e}$$

$$y_i = F_{s_{i+1}} - F_{s_i} \iff \vec{y} = \begin{bmatrix} -F_{s_1} + F_{s_2} + 0F_{s_3} \\ 0F_{s_1} - F_{s_2} + F_{s_3} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} F_{s_1} \\ F_{s_2} \\ F_{s_3} \end{bmatrix}$$

$$= -A^T C\vec{e}$$

$$= \boxed{-A^T C A \vec{u}(t)}$$

Student Work

c. (3 pts) Assume that the solution to this differential equation takes the form

$$\mathbf{u}(t) = \sin(\omega t + \phi)\mathbf{v}$$

for $\mathbf{v} \in \mathbb{R}^2$. Derive the corresponding eigenvalue problem using this ansatz.

$\vec{u}(t) = \sin(\omega t + \phi)\, v$

$= \sin(\omega t + \phi) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\Rightarrow \ddot{u}(t) = \frac{d^2}{dt^2} \begin{bmatrix} v_1 \sin(\omega t + \phi) \\ v_2 \sin(\omega t + \phi) \end{bmatrix}$

$\Rightarrow -\omega^2 \sin(\omega t + \phi) \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Side: $\frac{d}{dt} \sin(\omega t + \phi)$

$= \omega \cos(\omega t + \phi)$

$\frac{d^2}{dt^2} = -\omega^2 \sin(\omega t + \phi)$

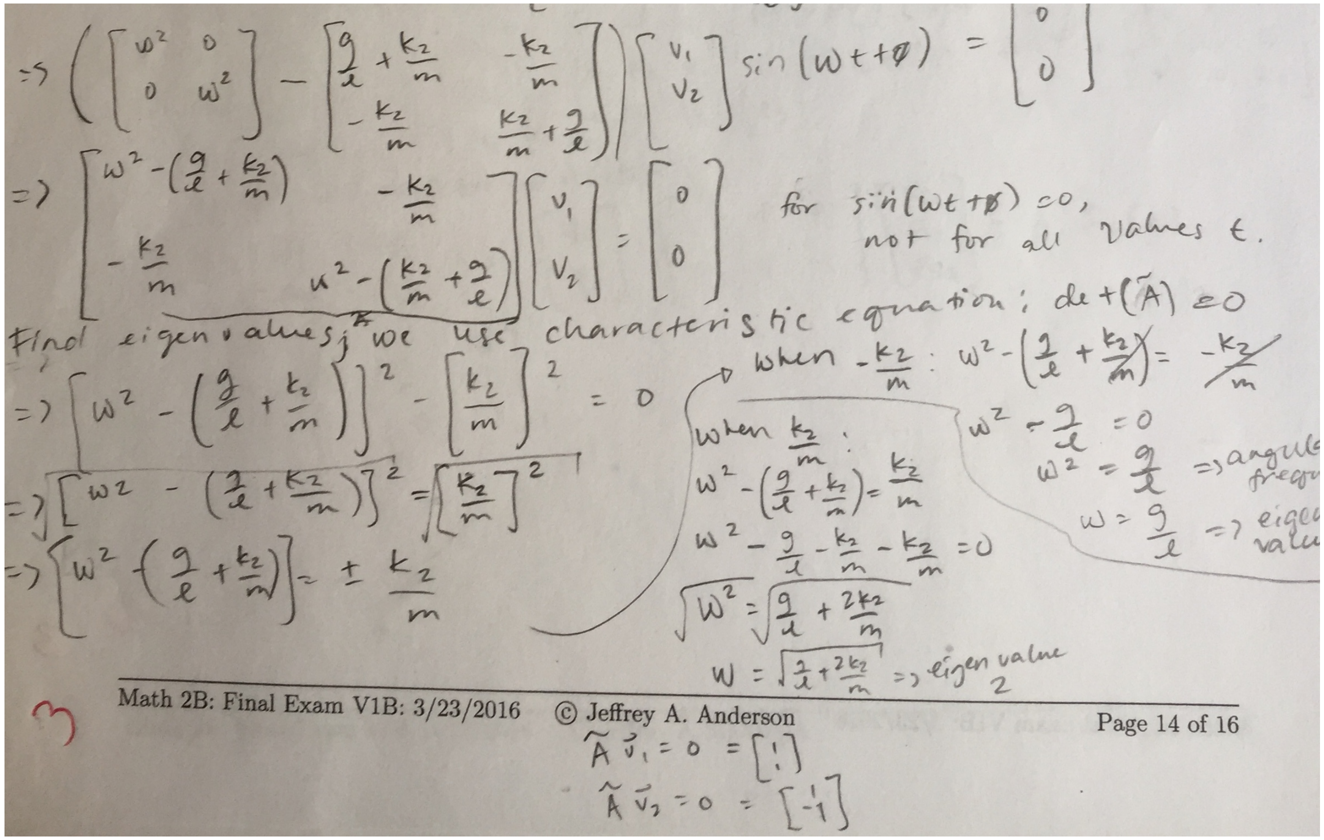
Since we know $m\ddot{u} + k\vec{u} = f(t)$,

$\Rightarrow -\omega^2 \sin(\omega t + \phi) \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} \frac{g}{l} + \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2}{m} + \frac{g}{l} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \sin(\omega t + \phi)$

$\Rightarrow \omega^2 \sin(\omega t + \phi) \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} \frac{g}{l} + \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2}{m} + \frac{g}{l} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \sin(\omega t + \phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \sin(\omega t + \phi) - \begin{bmatrix} \frac{g}{l} + \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2}{m} + \frac{g}{l} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \sin(\omega t + \phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Student Work



$$\Rightarrow \left(\begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} - \begin{bmatrix} \frac{g}{\ell} + \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2}{m} + \frac{g}{\ell} \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \sin(\omega t + \phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \omega^2 - \left(\frac{g}{\ell} + \frac{k_2}{m}\right) & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \omega^2 - \left(\frac{k_2}{m} + \frac{g}{\ell}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for $\sin(\omega t + \phi) = 0$, not for all values t.

Find eigenvalues, we use characteristic equation: $\det(\tilde{A}) = 0$

$$\Rightarrow \left[\omega^2 - \left(\frac{g}{\ell} + \frac{k_2}{m}\right)\right]^2 - \left[\frac{k_2}{m}\right]^2 = 0$$

$$\Rightarrow \sqrt{\left[\omega^2 - \left(\frac{g}{\ell} + \frac{k_2}{m}\right)\right]^2} = \sqrt{\left[\frac{k_2}{m}\right]^2}$$

$$\Rightarrow \left[\omega^2 - \left(\frac{g}{\ell} + \frac{k_2}{m}\right)\right] = \pm \frac{k_2}{m}$$

when $-\frac{k_2}{m}$: $\omega^2 - \left(\frac{g}{\ell} + \frac{k_2}{m}\right) = -\frac{k_2}{m}$

$\omega^2 - \frac{g}{\ell} = 0$

$\omega^2 = \frac{g}{\ell}$ ⇒ angula freq

$\omega = \frac{g}{\ell}$ ⇒ eigen valu

when $\frac{k_2}{m}$:

$\omega^2 - \left(\frac{g}{\ell} + \frac{k_2}{m}\right) = \frac{k_2}{m}$

$\omega^2 - \frac{g}{\ell} - \frac{k_2}{m} - \frac{k_2}{m} = 0$

$\sqrt{\omega^2} = \sqrt{\frac{g}{\ell} + \frac{2k_2}{m}}$

$\omega = \sqrt{\frac{g}{\ell} + \frac{2k_2}{m}}$ ⇒ eigen value 2

3

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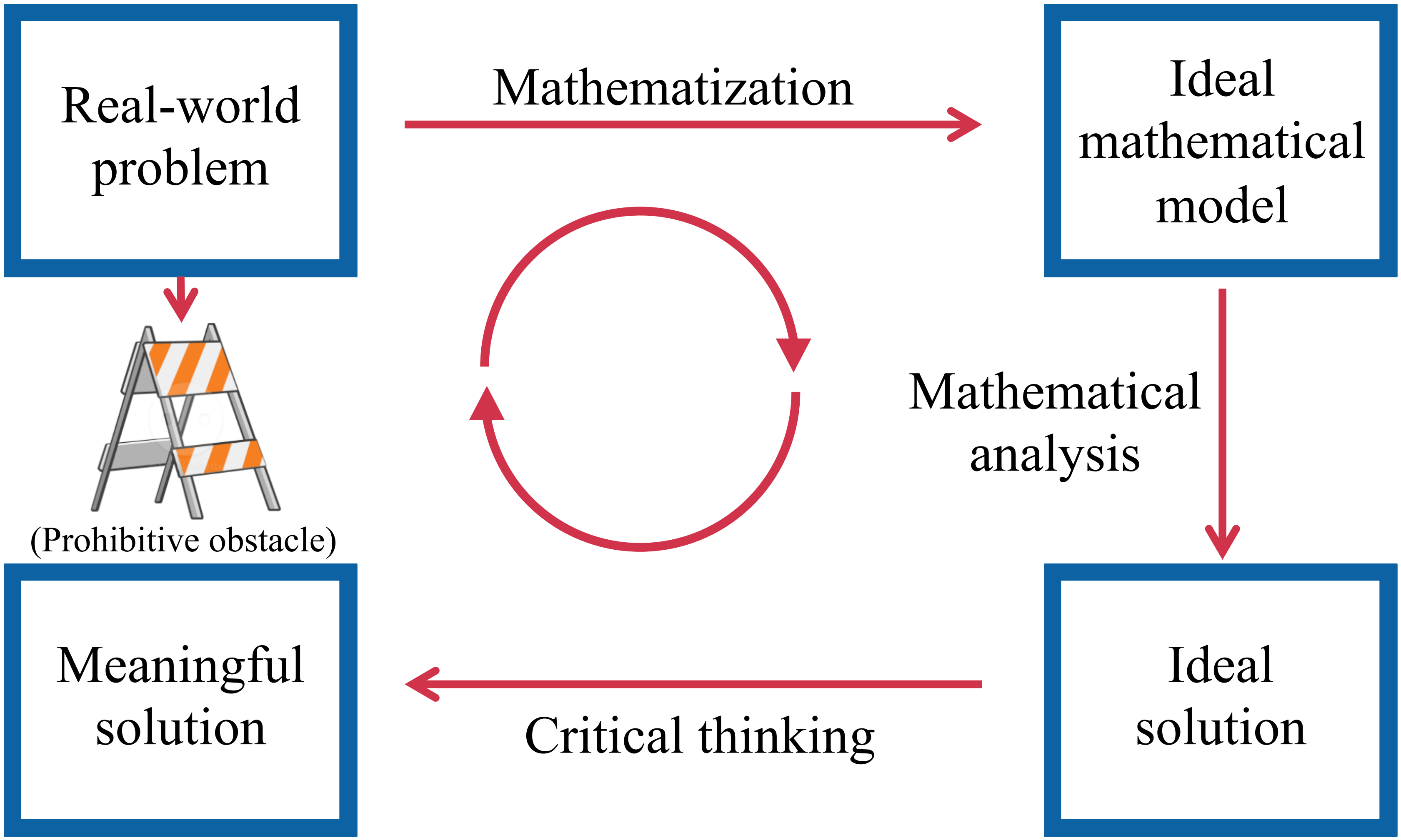
$\tilde{A}\vec{v}_1 = 0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\tilde{A}\vec{v}_2 = 0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The Mathematical Modeling Process



Questions

Your Feedback

Please work on back of survey

PART II: CURRENT WORK LOAD					
8. How interested are you in trying this eigenvalue modeling activity in your classroom?					
☐ 1 Not at all interested	☐ 2	☐ 3	☐ 4	☐ 5	☐ 6 Very interested
9. What resources do you think you would need to implement this activity in your classroom?					
10. What was your favorite part of this presentation?					