The Linear Systems Problem

Electrified

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Common Practices in Linear Algebra Education

Start with Gaussian Elimination

eg. Anton, Kolman & Hill, Larson, Lay, Leon, Meyer, Olver & Shakiban, Strang

Use Implicit Applications

A Better Way?

Start with Matrix Multiplication

Use Explicit Applications

A Better Way?
Four Major Problems

Matrix Multiplication Problem: $A\boldsymbol{x} = \boldsymbol{b}$

Linear Systems Problem: $A\boldsymbol{x} = \boldsymbol{b}$

Least Squares Problem: $\min_{\boldsymbol{x} \in \mathbb{R}^n} \|A\boldsymbol{x} - \boldsymbol{b}\|_2$

Eigenvalue Problem: $A\boldsymbol{x} = \lambda\boldsymbol{x}$

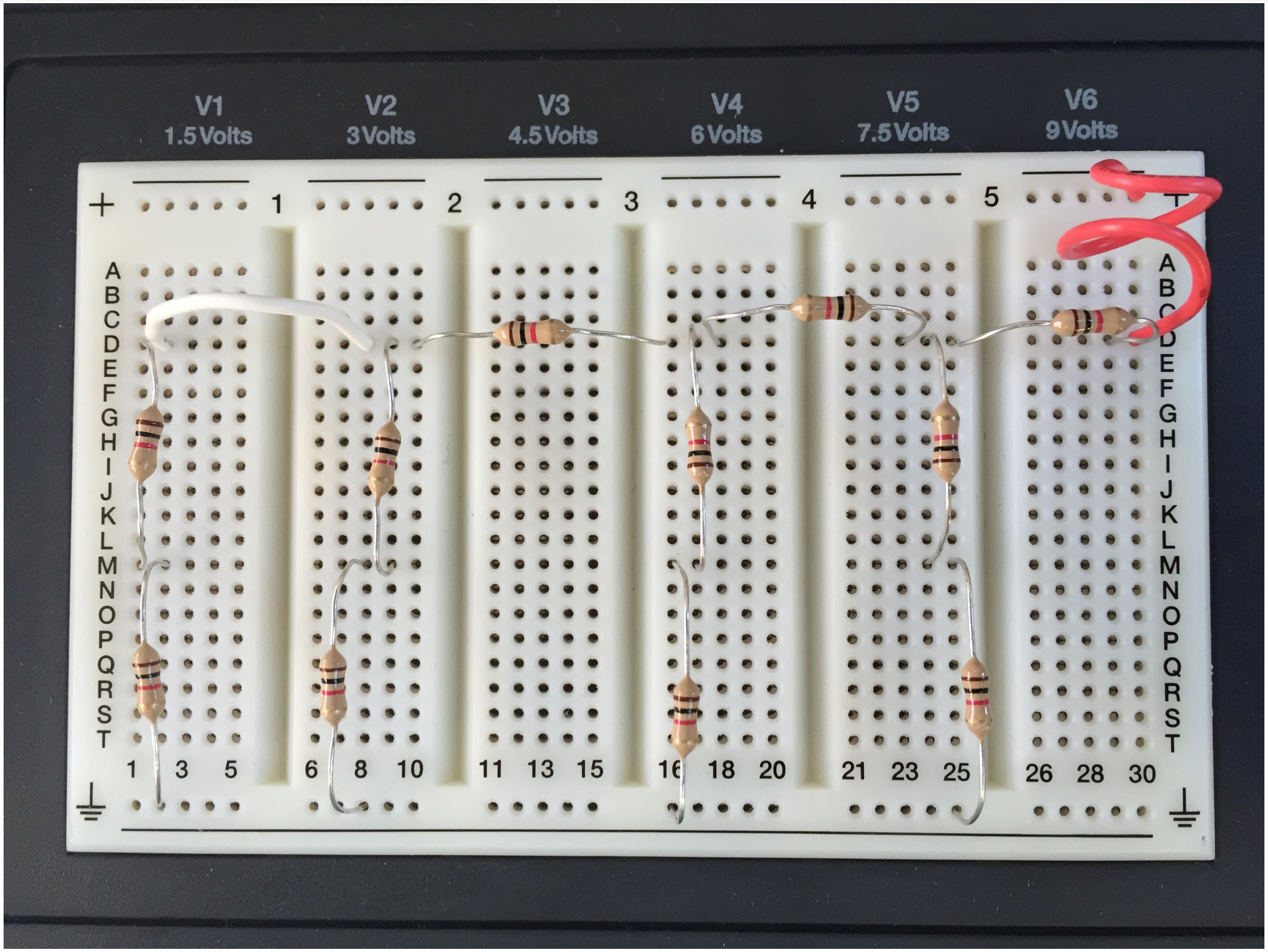
Explicit Applications: Linear Systems in Circuit Analysis

Linear Systems Problem: $A\mathbf{x} = \mathbf{b}$

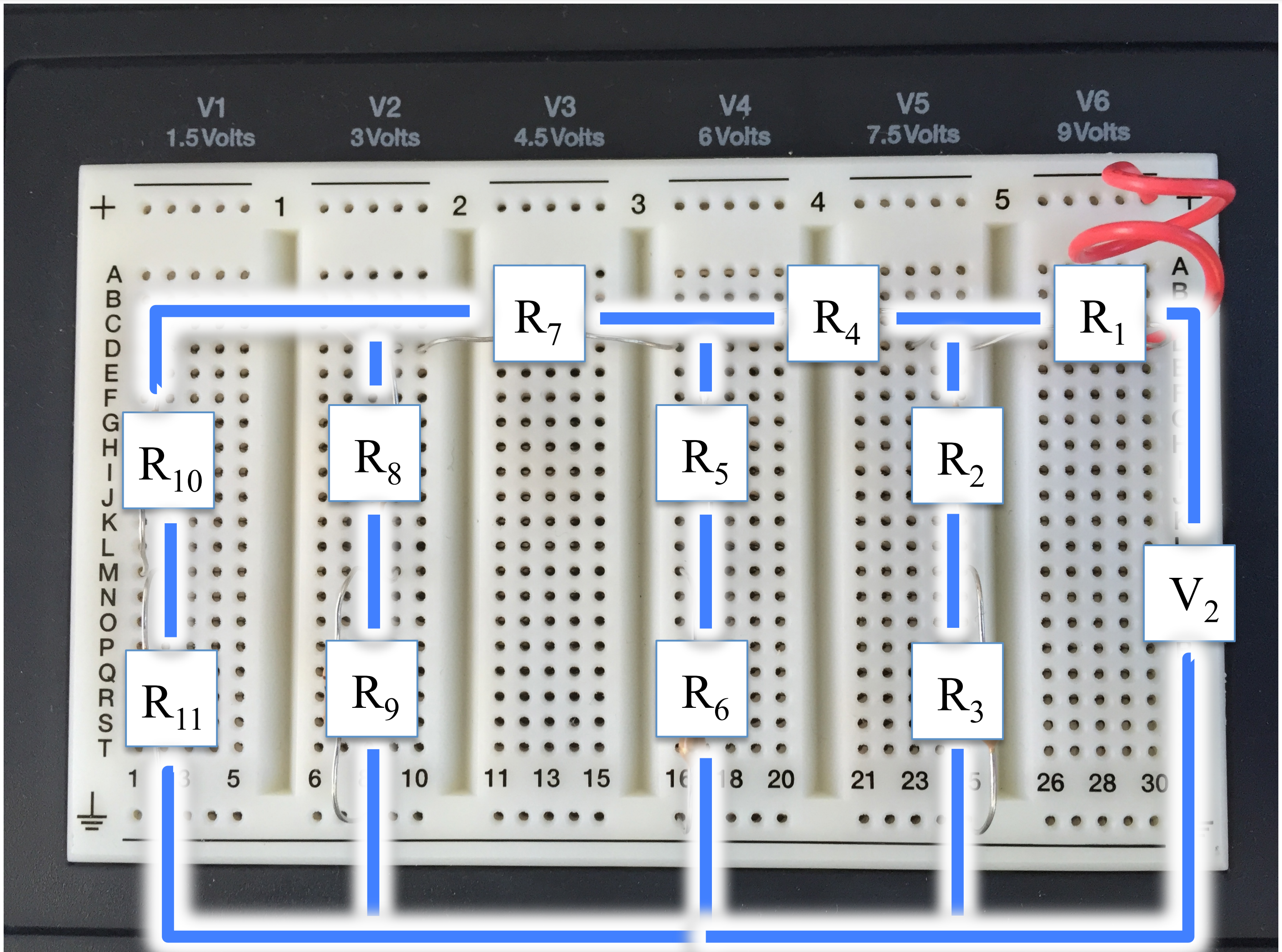
Use EXPLICIT Application:

Document Camera

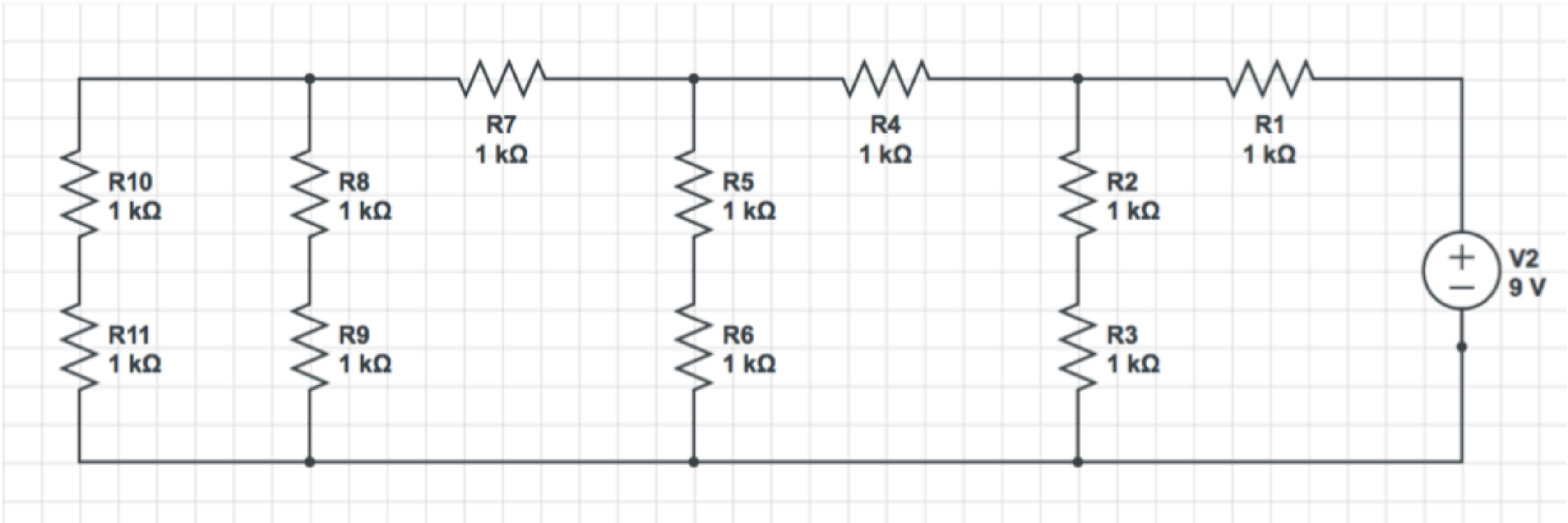
Resistor Ladder:
Ideal Circuit Model



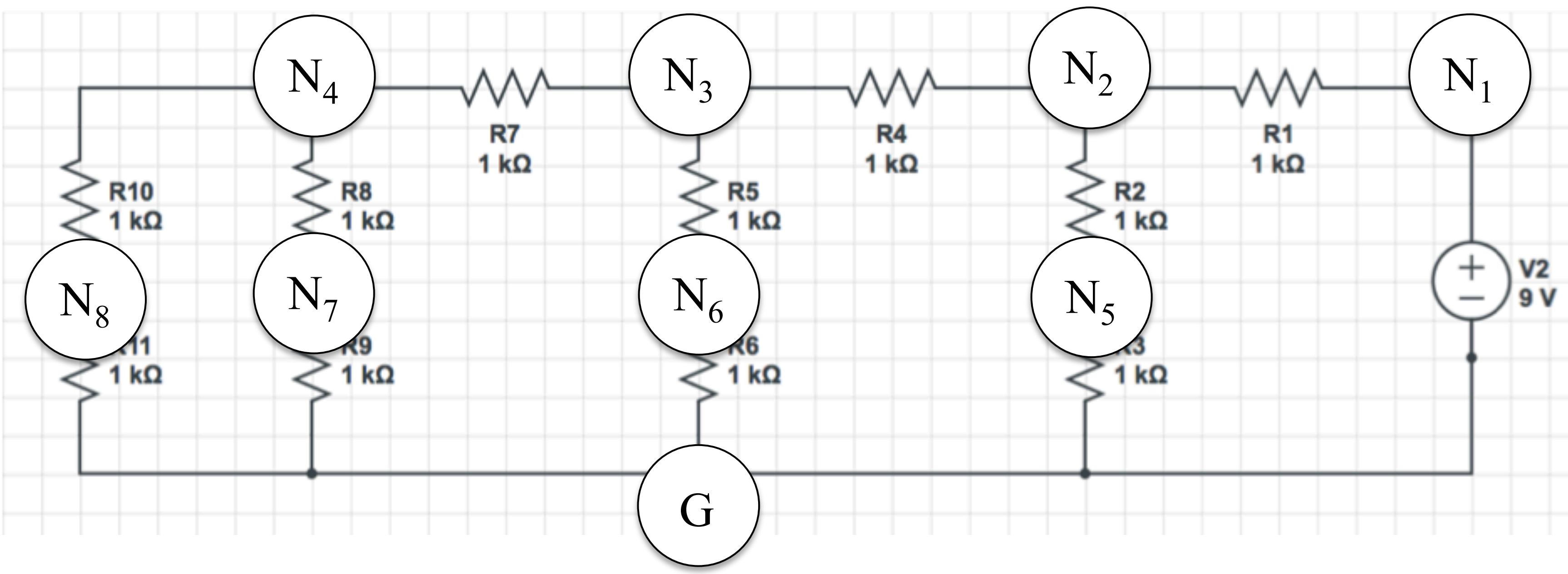
Resistor Ladder:
Ideal Circuit Model



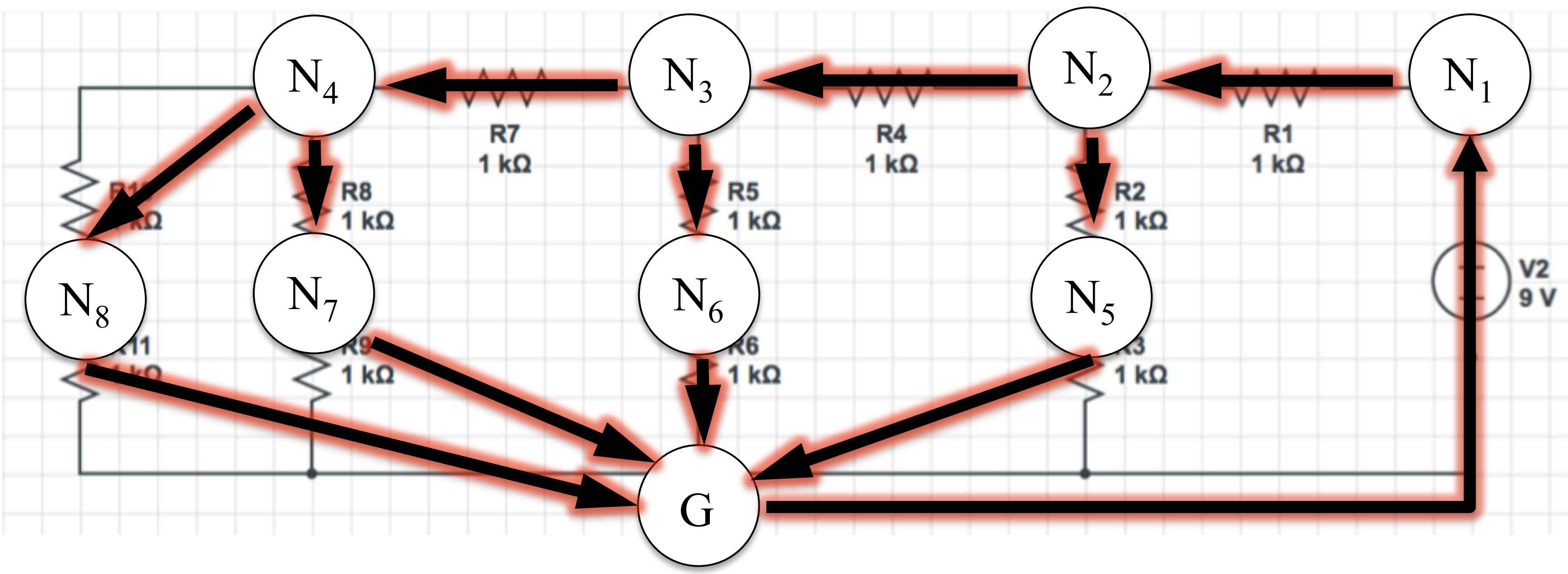
Resistor Ladders:
Directed Graph Model



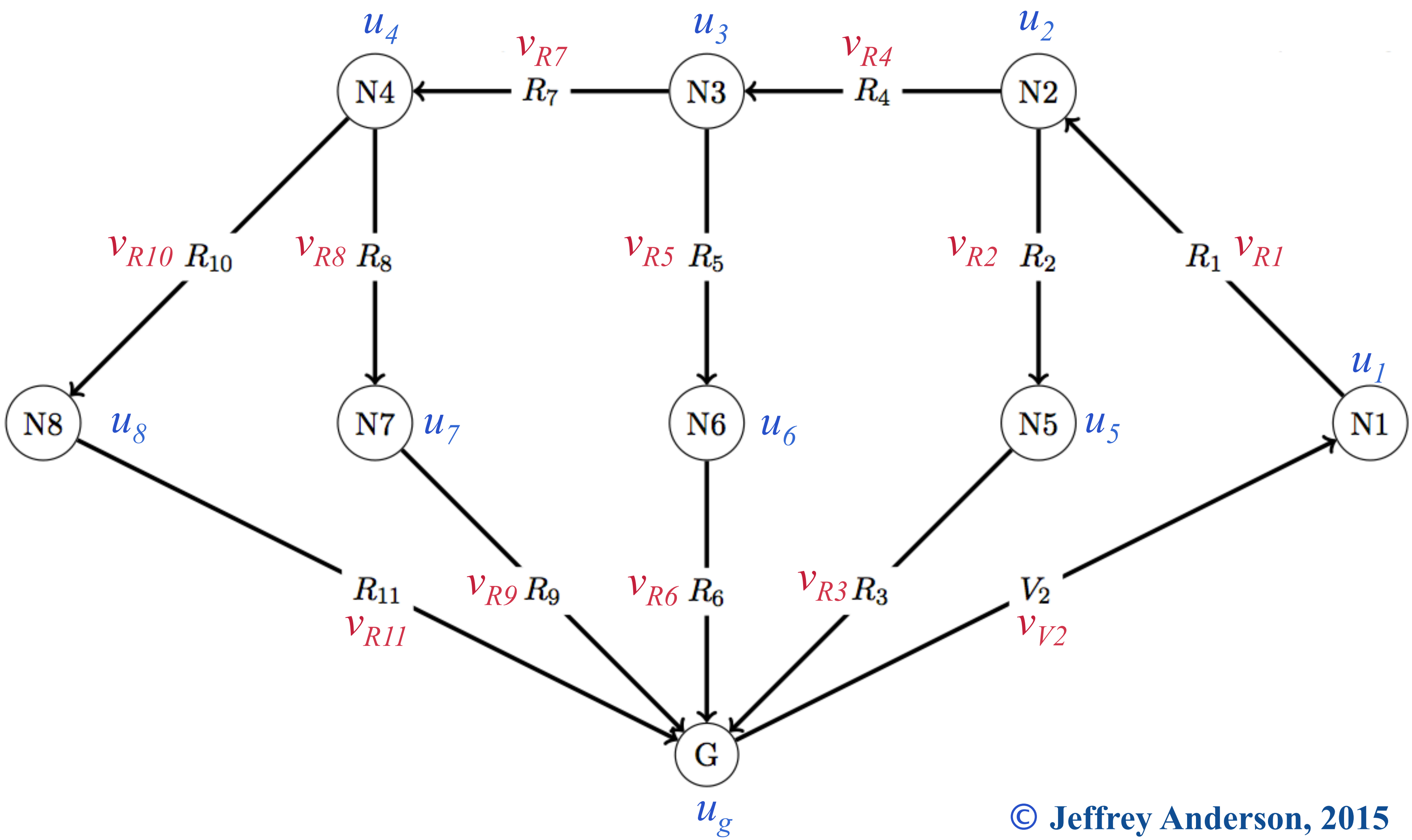
Resistor Ladders:
Directed Graph Model



Resistor Ladders: Directed Graph Model



Resistor Ladders: Voltage Calculations (KVLs)



Resistor Ladders:
KVLs and Matrix Multiplication

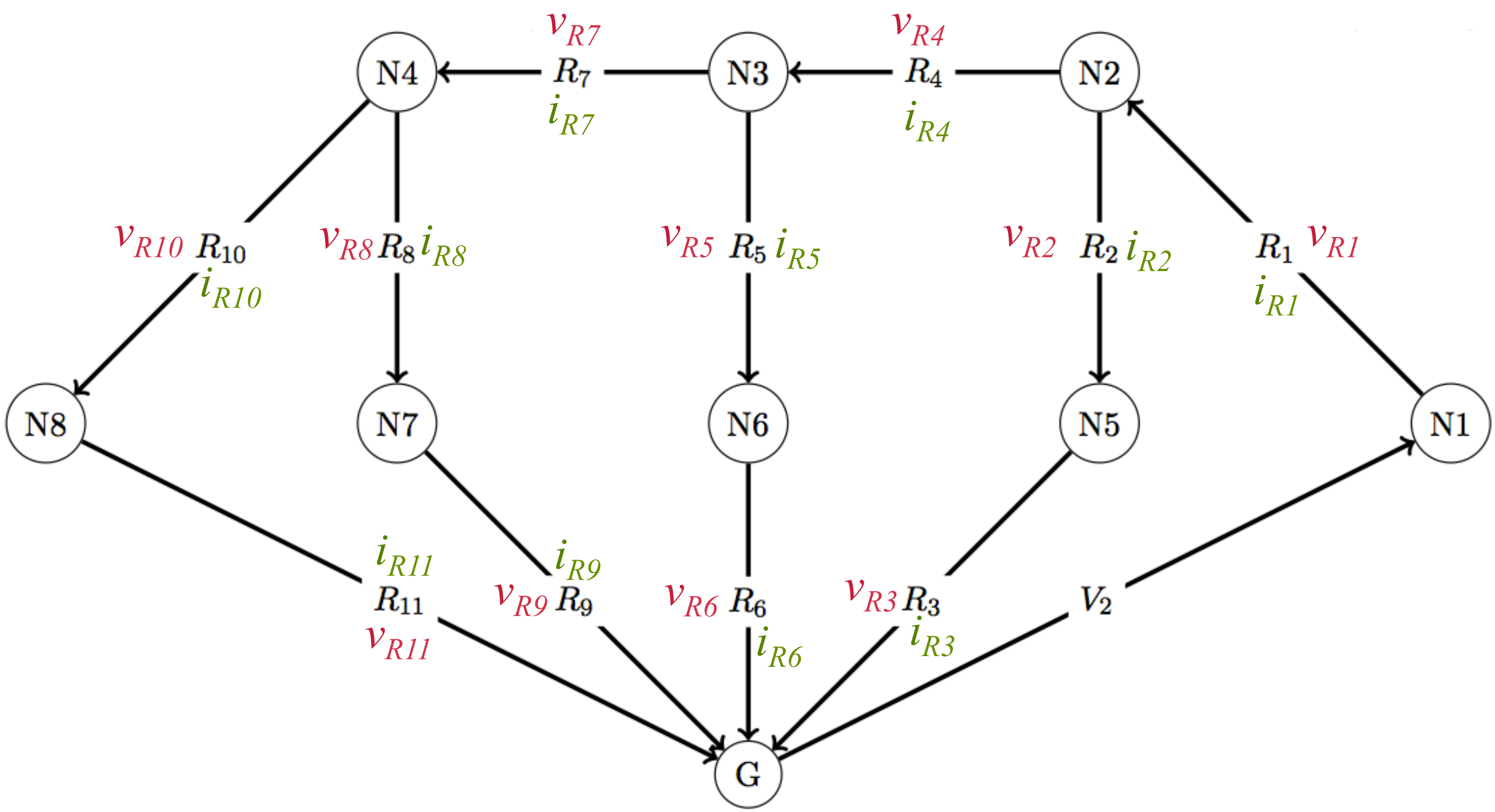
$$\begin{bmatrix} v_{V2} \\ v_{R1} \\ v_{R2} \\ v_{R3} \\ v_{R4} \\ v_{R5} \\ v_{R6} \\ v_{R7} \\ v_{R8} \\ v_{R9} \\ v_{R10} \\ v_{R11} \end{bmatrix} = \begin{bmatrix} u_g - u_1 \\ u_1 - u_2 \\ u_2 - u_5 \\ u_5 - u_g \\ u_2 - u_3 \\ u_3 - u_6 \\ u_6 - u_g \\ u_3 - u_4 \\ u_4 - u_7 \\ u_7 - u_g \\ u_4 - u_8 \\ u_8 - u_g \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_g \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}$$

Resistor Ladders:
KVLs and Matrix Multiplication

$$
\begin{bmatrix} v_{R1} \\ v_{R2} \\ v_{R3} \\ v_{R4} \\ v_{R5} \\ v_{R6} \\ v_{R7} \\ v_{R8} \\ v_{R9} \\ v_{R10} \\ v_{R11} \end{bmatrix} = \begin{bmatrix} u_1 - u_2 \\ u_2 - u_5 \\ u_5 - u_g \\ u_2 - u_3 \\ u_3 - u_6 \\ u_6 - u_g \\ u_3 - u_4 \\ u_4 - u_7 \\ u_7 - u_g \\ u_4 - u_8 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}
$$

$$\mathbf{v} = A^T \boldsymbol{u}$$

Resistor Ladders: BRC Calculations (Ohm's Law)

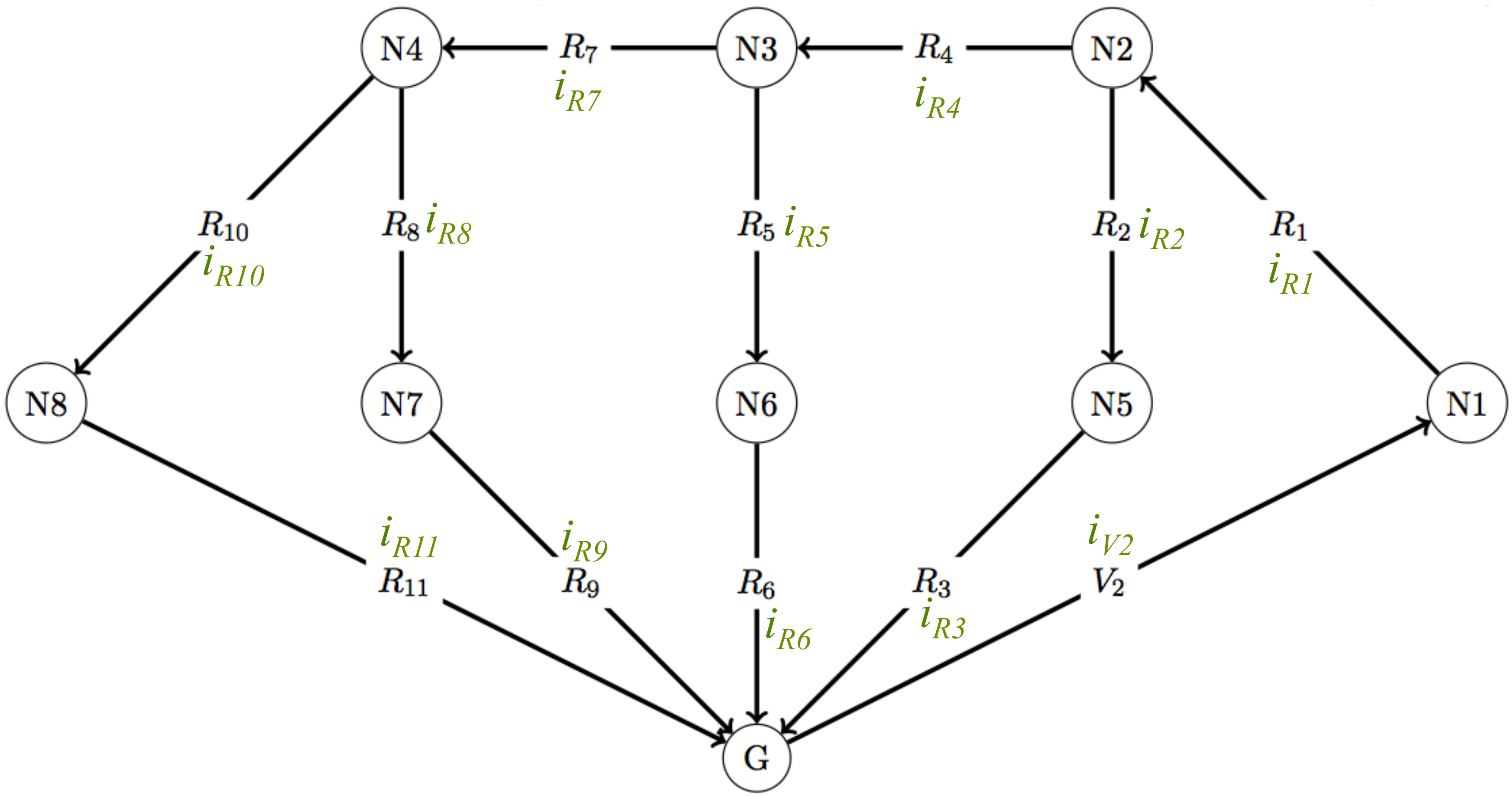


Resistor Ladders:
BCRs and Matrix Multiplication

$$\begin{bmatrix} v_{R1} \\ v_{R2} \\ v_{R3} \\ v_{R4} \\ v_{R5} \\ v_{R6} \\ v_{R7} \\ v_{R8} \\ v_{R9} \\ v_{R10} \\ v_{R11} \end{bmatrix} = \begin{bmatrix} R_1 \cdot i_1 \\ R_2 \cdot i_2 \\ R_3 \cdot i_3 \\ R_4 \cdot i_4 \\ R_5 \cdot i_5 \\ R_6 \cdot i_6 \\ R_7 \cdot i_7 \\ R_8 \cdot i_8 \\ R_9 \cdot i_9 \\ R_{10} \cdot i_{10} \\ R_{11} \cdot i_{11} \end{bmatrix} = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{11} \end{bmatrix} \begin{bmatrix} i_{R1} \\ i_{R2} \\ i_{R3} \\ i_{R4} \\ i_{R5} \\ i_{R6} \\ i_{R7} \\ i_{R8} \\ i_{R9} \\ i_{R10} \\ i_{R11} \end{bmatrix}$$

$$\mathbf{v} = R\boldsymbol{i}$$

Resistor Ladders: Current Calculations (KCLs)



Resistor Ladders: KCLs and Matrix Multiplication

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
=
\begin{bmatrix}
i_{R1} \\
-i_{R1} + i_{R2} + i_{R4} \\
-i_{R4} + i_{R5} + i_{R7} \\
-i_{R7} + i_{R8} + i_{R10} \\
-i_{R2} + i_{R3} \\
-i_{R5} + i_{R6} \\
-i_{R8} + i_{R9} \\
-i_{R10} + i_{R11}
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
i_{R1} \\ i_{R2} \\ i_{R3} \\ i_{R4} \\ i_{R5} \\ i_{R6} \\ i_{R7} \\ i_{R8} \\ i_{R9} \\ i_{R10} \\ i_{R11}
\end{bmatrix}
$$

$$\mathbf{0} = A\boldsymbol{i}$$

Resistor Ladders: All Together Now

$$\mathbf{v} = A^T \boldsymbol{u}$$

$$\mathbf{v} = R\boldsymbol{i}$$

$$\mathbf{0} = A\boldsymbol{i}$$

$$\mathbf{0} = AR^{-1} A^T \mathbf{u}$$

Resistor Ladders: All Together Now

$$\mathbf{v} = A^T \boldsymbol{u}$$

$$R^{-1}\mathbf{v} = \boldsymbol{i}$$

$$\mathbf{0} = A\boldsymbol{i}$$

$$\mathbf{0} = AR^{-1} A^T \mathbf{u}$$