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Math 48A, Lesson 16: Quadratic Functions

1. Make Connections between the Two Forms of a Quadratic Function

Consider the two forms for a quadratic function:

$$f(x) = \boxed{a\,x^2 + b\,x + c}$$

Standard form
of quadratic function

$$f(x) = \boxed{a\,(x-h)^2 + k}$$

Vertex form
of quadratic function

Look back on our Lesson 15 handout. Write equations for the values of h, and k in terms of the values of a, b, and c. Write two different algebraic approaches you can use to discover this connection.

Let's compare our two forms

$$ax^2 + bx + c = a\cdot(x-h)^2 + K$$

Let's explore the relationship:

$$a\cdot \underset{\text{perfect square}}{\boxed{(x-h)^2}} + K$$

$$= a\cdot(\underbrace{x^2 - 2xh + h^2}) + K$$

Side Note:

$$(x-h)^2 = (x-h)\cdot(x-h)$$

$$= x\cdot(x-h) - h\cdot(x-h)$$

$$= x^2 - xh - xh + h^2$$

$$= x^2 - 2xh + h^2$$

$$= a\cdot(x^2 - 2xh + h^2) + K$$

$$= a\cdot x^2 - 2\cdot a\cdot h\cdot x + a h^2 + K$$

$$= a\, x^2 + \underbrace{-2\cdot a\cdot h}\cdot x + \underbrace{K + a\cdot h^2}$$

$$= a x^2 + b x + c$$

When we compare, we notice

$$b = \boxed{-2 \cdot a \cdot h} \Rightarrow \frac{-\cancel{2} \cdot \cancel{a} \cdot h}{-\cancel{2} \cdot \cancel{a}} = \frac{b}{-2 \cdot a}$$

$$\Rightarrow \boxed{h = \frac{b}{-2 \cdot a}}$$

$$c = \boxed{K + a h^2} \Rightarrow K = c - a \cdot h^2$$

$$\Rightarrow K = c - a \cdot \left[\frac{b}{-2 \cdot a}\right]^2$$

$$\Rightarrow K = c - a \cdot \left[\frac{b^2}{4 \cdot a^2}\right]$$

see side note below

side note:

$$\left[\frac{b}{-2\cdot a}\right]^2 = \left[\frac{b}{-2\cdot a}\right]\cdot\left[\frac{b}{-2a}\right]$$

$$= \frac{b\cdot b}{-2\cdot a\cdot -2\cdot a}$$

$$= \frac{b^2}{4\cdot a^2}$$

$$\Rightarrow \boxed{K = c - a\cdot\frac{b^2}{4\cdot a^2} = c - \frac{b^2}{4\cdot a}}$$

side note:

$$a\cdot\frac{b^2}{4\cdot a^2} = \frac{a}{1}\cdot\frac{b^2}{4\cdot a\cdot a}$$

$$= \frac{\cancel{a}\cdot b^2}{1\cdot\cancel{a}\cdot a\cdot 4}$$

$$= \frac{b^2}{4\cdot a}$$

so from all this algebra, we see

standard form $\quad\quad$ vertex form

$$\boxed{a x^2 + bx + c} = \boxed{a \cdot (x - h)^2 + K}$$

we have

$$\boxed{h = \frac{b}{-2 \cdot a}} \, , \quad \boxed{K = c - \frac{b^2}{4 \cdot a}}$$

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2. Practice Identifying the Two Forms of a Quadratic Function

Write each of functions below in BOTH standard and vertex form. In other words, specifically identify the values of a, b, and c as well as the values of a, h, and k.

2A. $f(x) = 5x^2 - 30x + 49$

2B. $g(x) = -x^2 + x - 2$

2C. $h(x) = 4 - x^2$

2D. $j(x) = 2x^2 + 4x - 6$

2E. $k(x) = -2x^2 + 12x - 16$

Problem 2D

Let $j(x) = 2x^2 + 4x - 6$

$ax^2 + bx + c$

$a = 2$, $b = 4$, $c = -6$

Let's translate into vertex form

$$\boxed{h = \frac{b}{-2\cdot a} = \frac{4}{-2\cdot 2} = \frac{4}{-4} = -1}$$

$$k = c - \frac{b^2}{4\cdot a} = -6 - \frac{[4]^2}{4\cdot(2)}$$

$$= -6 - \frac{16}{8}$$

$$= -6 - 2$$

$$\Rightarrow \boxed{K = -8}$$

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$\boxed{\text{Problem 2D}}$

Let $j(x) = \boxed{2x^2 + 4x - 6}$ standard form

$= 2\cdot(x - -1)^2 + -8$

$= \boxed{2\cdot(x+1)^2 - 8}$ vertex form

Problem 2A

Let $f(x) = 5x^2 - 30x + 49$

$= 5x^2 + -30x + 49$

$ax^2 + bx + c$

$a = 5, \quad b = -30, \quad c = 49$

Let's translate into vertex form

$$h = \frac{b}{-2 \cdot a} = \frac{-30}{-2 \cdot 5} = \frac{-30}{-10} = 3$$

$$k = c - \frac{b^2}{4a} = 49 - \frac{[-30]^2}{4 \cdot 5}$$

$$= 49 - \frac{900}{20}$$

$$= 49 - 45$$

$$= 4$$

Problem 2A

$\Rightarrow f(x) = \boxed{5x^2 - 30x + 49}$ standard form

$= \boxed{5(x-3)^2 + 4}$ vertex form

Problem 2B

Let $g(x) = -x^2 + x - 2$

$= -1 \cdot x^2 + 1 \cdot x + -2$

$ax^2 + bx + c$

$a = -1 \quad , \quad b = 1 \quad , \quad c = -2$

Let's translate into vertex form

$$h = \frac{b}{-2a} = \frac{1}{-2 \cdot (-1)} = \frac{1}{2}$$

$$K = c - \frac{b^2}{4a} = -2 - \frac{1^2}{4 \cdot (-1)}$$

$$= -2 - \frac{1}{-4}$$

$$= -2 + \frac{1}{4}$$

$$\Rightarrow \quad K = -\frac{7}{4}$$

Problem 2B

$$\Rightarrow g(x) = \boxed{-x^2 + x - 2}$$ standard form

$$= \boxed{-\left(x - \frac{1}{2}\right)^2 - \frac{7}{4}}$$ vertex form

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Problem 2C

$$h(x) = -x^2 + 4$$

$$= \boxed{-1 \cdot x^2 + 0 \cdot x + 4}$$ ← standard form

$$= \boxed{-(x-0)^2 + 4}$$ ← vertex form

Problem 2E

$$K(x) = \boxed{-2x^2 + 12x + -16}$$ ← standard form

$$= \boxed{-2(x-3)^2 + 2}$$ ← vertex form

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3B. Express $j(x) = 2x^2 + 4x - 6$ in vertex form.

$$j(x) = 2x^2 + 4x - 6$$

k-value

$$= 2(x - -1)^2 + -8$$

h-value

3C. What is the minimum value of our function? What is the range of $j(x)$?

- Minimum value is $y = -8 = k$

  produced by input $x = -1 = h$

  $\Rightarrow f(h) = K$ is the minimum

- We Know the range is $y \geq -8$

  $[-8, \infty)$

  since $a = 2 > 0$

$$K(x) = -2x^2 + 12x - 16$$

$a = -2, \quad b = 12, \quad c = -16$

Let's translate into vertex form

$$h = \frac{b}{-2\cdot a} = \frac{12}{-2\cdot(-2)} = \frac{12}{4} = 3$$

$$k = c - \frac{b^2}{4\cdot a} = -16 - \frac{12^2}{4\cdot(-2)}$$

$$= -16 - \frac{144}{-8}$$

$$= -16 + 18$$

$$= 2$$

pants down

$$\Rightarrow K(x) = -2\cdot(x-3)^2 + 2$$

$h = 3, \quad k = 2$

$\boxed{\text{vertex @ } (3, 2)}$

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3. Graphical Interpretation of Vertex Form

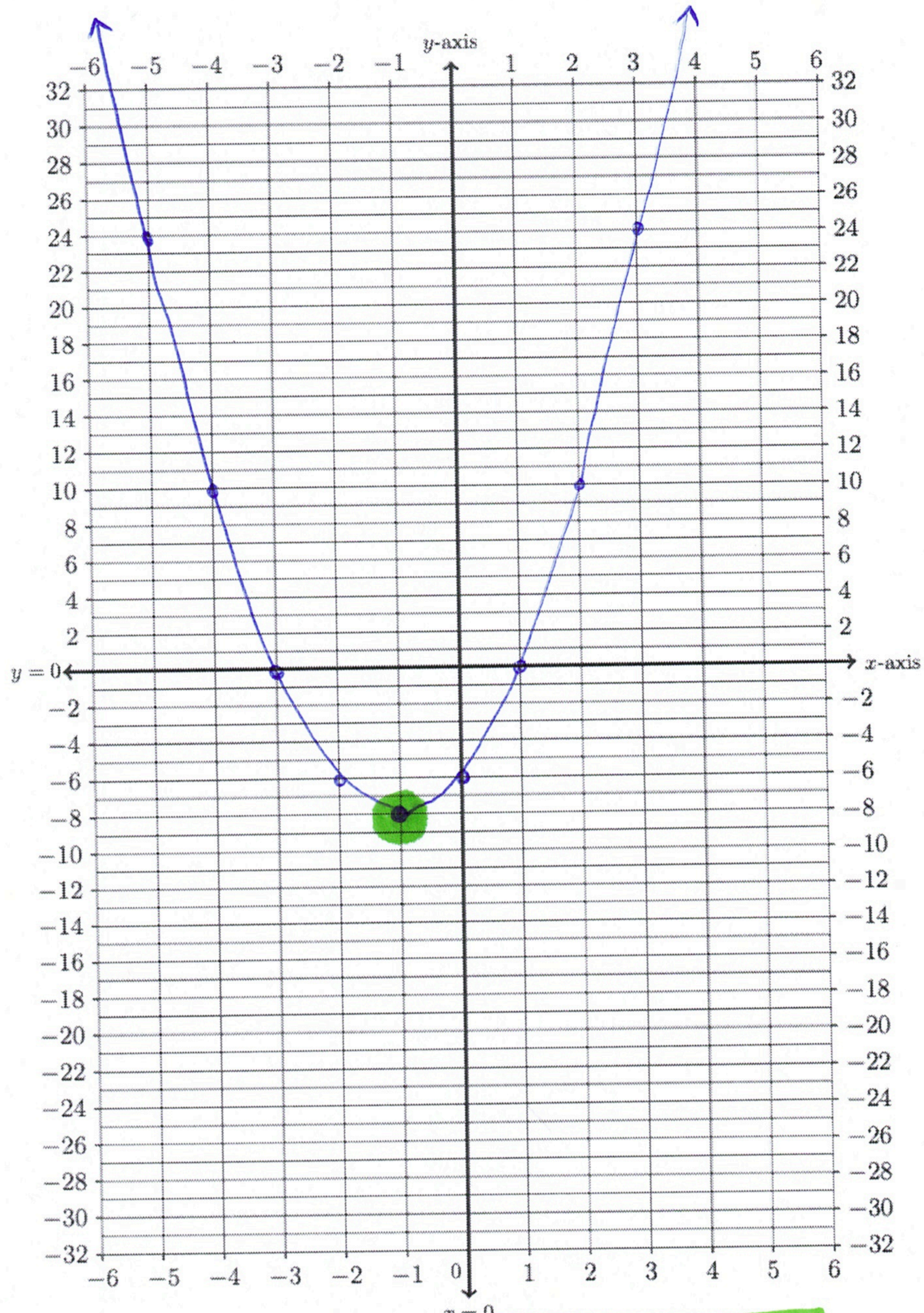
3A. Consider the standard form for a quadratic function:

$$j(x) = \boxed{2x^2 + 4x - 6} = \boxed{2(x+1)^2 - 8}$$

standard form   vertex form

Sketch a graph of this function on the axis below.

Input	*Output*
x	$j(x)$
−6	42
−5	24
−4	10
−3	0
−2	-6
−1	-8
0	-6
1	0
2	10
3	24
4	42
5	64
6	90



vertex : $(h, k) = (-1, -8)$

↑

this is a minimum

$f(-1) = -8$

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3D. What connections do you see between the vertex form of our function

$$j(x) = 2\,x^2 + 4\,x - 6$$

and the corresponding graph of the parabola that you created in problem 3A.

Looking at the vertex form

$$j(x) = \overbrace{2}^{\text{parabola points upward since } a>0 \text{ with double the rate as } x^2}\,(x - \underbrace{-1}_{\text{1st-coordinate for the vertex of the parabola}})^2 + \overbrace{-8}^{\text{2nd-coordinate for the vertex of the parabola}}$$

The vertex is at point (-1, -8).

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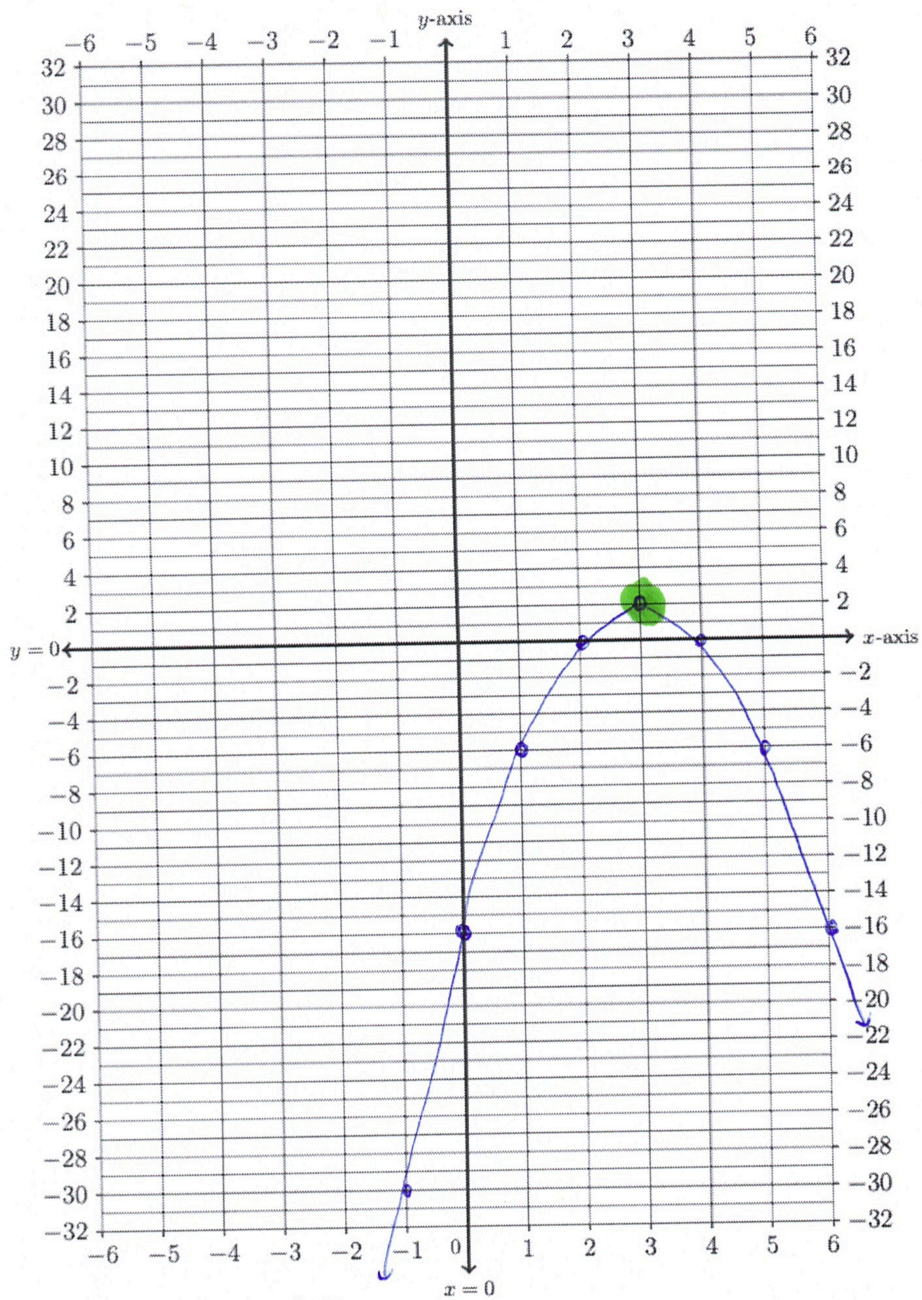
4. Graphical Interpretation of Vertex Form

4A. Consider the standard form for a quadratic function:

$$k(x) = -2\,x^2 + 12\,x - 16$$

Sketch a graph of this function on the axis below.

Input	*Output*
x	$k(x)$
-6	-160
-5	-126
-4	-96
-3	-70
-2	-48
-1	-30
0	-16
1	-6
2	0
3	2
4	0
5	-6
6	-16



vertex at (3,2)

max value @ vertex

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4B. Express $k(x) = -2x^2 + 12x - 16$ in vertex form.

$$K(x) = -2x^2 + 12x - 16$$

K-value

$$= -2(x - \underbrace{3}_{\text{h-value}})^2 + 2$$

h-value

4C. What is the max value of our function? What is the range of $k(x)$?

- Maximum value is the output y-value of our vertex given as

$$k = c - \frac{b^2}{4a} = 2$$

- The input value $x = h = \frac{b}{-2a}$ produces the maximum with $f(h) = k$.

- We see the range is $y \le 2$ which, in interval form, is given as $(-\infty, 2]$ since parabola points down with $a = -2 < 0$.

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4D. What connections do you see between the vertex form of our function

$$k(x) = -2\,x^2 + 12\,x - 16$$

and the corresponding graph of the parabola that you created in problem 4A.

Looking at the vertex form, we see

$$k(x) = \overbrace{-2}^{\text{parabola points downward since } a<0 \text{ with double the rate as } x^2}(x - \underbrace{3}_{\text{1st-coordinate for the vertex of the parabola}})^2 + \overbrace{2}^{\text{2nd-coordinate for the vertex of the parabola}}$$

The vertex is at point $(3, 2)$.

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5. Use Vertex Form to Find Maximum or Minimum Values

Take a look at the image below.

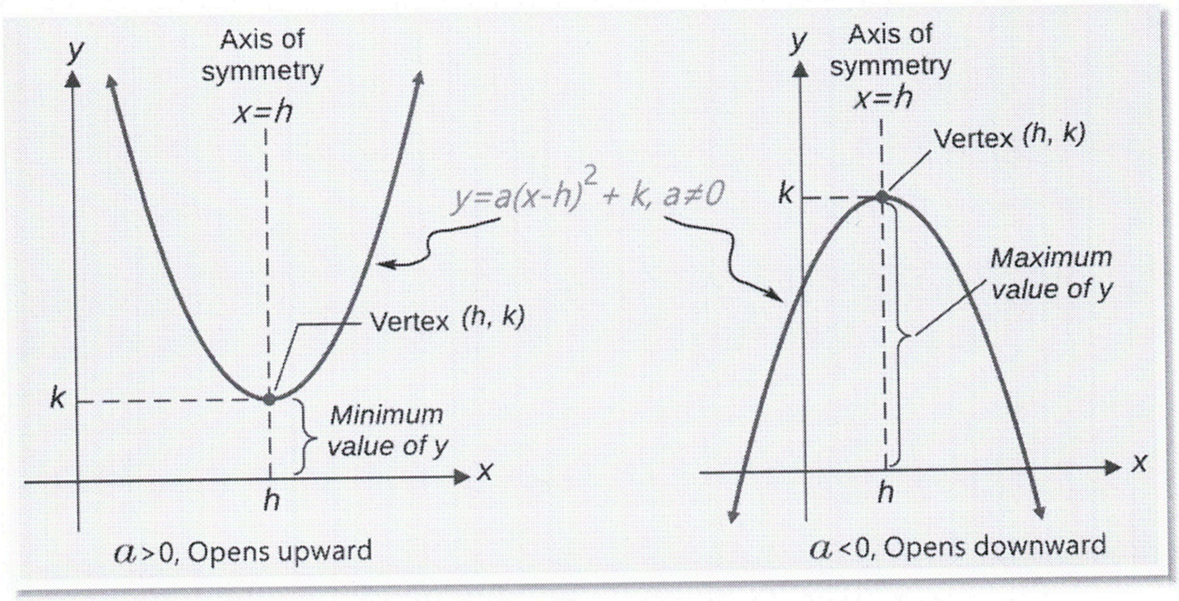

Image Source: https://www.targetmathematics.com/2020/06/g10-chap5-quadratic-functions-exercise-52.html

For each parabola that you see, explain using simple abuelita language what this image is saying. Make connections between the image that you see here and the work you did on Problems 1 – 4 in this Lesson 16 worksheet.

The vertex form of a quadratic function is given as

$$f(x) = a(x-h)^2 + K$$

We can use the vertex form to identify the location of the vertex, which is at the point (h, K).

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6. Formula for Maximum or Minimum Values

Look back at the work you did on Problems 1 and 2 of this Lesson 16 handout. Come up with a general formula to find the minimum or maximum value of a quadratic function in the form

$$f(x) = a\,x^2 + b\,x + c$$

We recall that

$$f(x) = a x^2 + bx + c$$

$$= a(x - h)^2 + K$$

where $h = \dfrac{b}{-2a}$ and $K = c - \dfrac{b^2}{4a}$.

The maximum value is the output value at the vertex given by

$$K = c - \frac{b^2}{4a}$$

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7. Use Your Formula for Maximum or Minimum Values

Use the formula you generated in Problem 6 above to find the minimum or maximum value for each of the quadratic functions below. Be sure you can explain your steps. Do your best to make connections between the max/min values you find in this problem and the vertex forms for these quadratic functions you found in Problem 2 from this Lesson 16 handout.

7A. $f(x) = 5x^2 - 30x + 49$, $a = 5$, $b = -30$, $c = 49$

7B. $g(x) = -x^2 + x - 2$, $a = -1$, $b = 1$, $c = -2$

7C. $h(x) = 4 - x^2$, $a = -1$, $b = 0$, $c = 4$

Problem 7A

For $f(x) = 5x^2 - 30x + 49$

$$= 5(x-3)^2 + 4$$

The coefficient $a > 0$ means the parabola points upward and thus function has a minimum value of

$$\boxed{K = c - \frac{b^2}{4a} = 4}$$

Problem 7B

For $g(x) = -x^2 + x - 2$

$$= -\left(x - \frac{1}{2}\right)^2 - \frac{7}{4}$$

The coefficient $a < 0$ means the parabola points downward and thus the function has a maximum value of

$$\boxed{K = c - \frac{b^2}{4a} = -\frac{7}{4}}$$

Problem 7C

For $h(x) = 4 - x^2$

$$= -(x - 0)^2 + 4$$

The coefficient $a < 0$ means we have a local max value of

$$\boxed{K = c - \frac{b^2}{4a} = 4}$$