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Math 48A, Lesson 15: Quadratic Functions

1. Find the Standard Form of a Quadratic Function

Consider the standard form for a quadratic function:

$$f(x) = a\,x^2 + b\,x + c$$

Put each of functions below in standard form. In other words, specifically identify the values of a, b, and c.

✓ 1A. $f(x) = 5\,x^2 - 30\,x + 49$

1B. $g(x) = -x^2 + x - 2$

✓ 1C. $h(y) = 4 - y^2$

✓ 1D. $j(w) = -5 + 3w$

✓ 1E. $k(t) = t^2 + 4t$

Let's consider:

variable squared

variable

1A. $f(x) = 5x^2 - 30x + 49$

$a\,x^2 + b\,x + c$

$= 5x^2 + -30x + 49$

$a = 5$, $b = -30$, $c = 49$

1E: $K(t) = t^2 + 4t$

CONSTANT

- the number in front of the variable to the zeroth power
- term with no variable
- just number with no variable after

$$at^2 + bt + c$$

the number in front of the variable squared

variable squared

the number in front of variable to the first power

$$= 1 \cdot t^2 + 4t^1 + 0 \cdot t^0$$

$a = 1$ $b = 4$ $c = 0$

(Recall: $t^0 = 1$)

1C. $h(y) = 4 - y^2$

$$= -1 \cdot y^2 + 0y + 4$$

$$ay^2 + by + c$$

$a = -1$, $b = 0$, $c = 4$

variable name   variable

1D. $j(w) = -5 + 3w$

$= 0w^2 + 3w + -5$

$aw^2 + bw + c$

$a = 0, \quad b = 3, \quad c = -5$

1B. $g(x) = -x^2 + x - 2$

$ax^2 + bx + c$

$= -1 \cdot x^2 + 1 \cdot x + -2$

$a = -1, \quad b = 1, \quad c = -2$

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2. Explore the Standard Form of a Quadratic Function

Consider the standard form for a quadratic function:

$$f(x) = a\,x^2 + b\,x + c$$

Using simple language, identify the role of each individual part of this function. Do your best to come up with descriptions for each of the following: x, x^2, a, b, and c.

the number in front of the variable alone

OR

the number in front of the variable to first power

the number in front of the variable squared

$$f(x) = \boxed{a}\, x^2 + \boxed{b}\, x + \boxed{c}$$

variable we use (inside the parenthesis)

the number without a variable

$$= a_2 x^2 + a_1 x^1 + a_0 \underbrace{x^0}_{1}$$

This fits into a more general pattern:

$$P_2(x) = a_2 x^2 + a_1 x^1 + a_0 x^0$$

$$P_3(x) = a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0 x^0$$

$$P_4(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0 x^0$$

These are known as polynomial functions.

We can generalize to an nth degree polynomial for any $n = 1, 2, 3, 4, \ldots$ given by

$$p_n(x) = a_n x^n + \cdots + a_1 x^1 + a_0 x^0$$

Quadratic functions are 2nd degree polynomials with $n = 2$.

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3. Explore the anatomy of perfect-square trinomials

Expand each of the following perfect squares and write as a trinomial in the form

$$x^2 + b\,x + c.$$

Show your steps and specifically identify values for coefficients b and c. The first one is done for you.

3A. $\boxed{(x-4)^2}$ perfect square : a single base to the second power

base: $(x-4)$

Let's consider the perfect square $(x-4)^2$:

$$(x-4)^2 = (x-4)\cdot(x-4)$$

$$= x\cdot(x-4) - 4\cdot(x-4)$$

$$= x^2 \underbrace{- 4x - 4x}_{} + 16$$

$$= x^2 - 8x + 16 \qquad b = -8, c = 16$$

$a=1, \quad b=-8, \quad c=16$

3B. $(x+3)^2$

perfect square

$$\boxed{(x+3)^2} = (x+3)\cdot(x+3)$$

distribute

$$= x\cdot(x+3) + 3\cdot(x+3)$$

$$= x^2 + 3x + 3x + 9$$

combine like terms

$$= \boxed{x^2 + 6x + 9}$$ ← trinomial

tri = three

nomial = term

$a=1, \quad b=6, \quad c=9$

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3C. $(x+11)^2$

$$= (x+11)\cdot(x+11)$$

$$= x\cdot(x+11) + 11\cdot(x+11)$$

$$= x^2 + 11x + 11x + 121$$

$$= x^2 + 22x + 121$$

$a=1$, $b=22$, $c=121$

3D. $\boxed{\left(x-\frac{7}{2}\right)^2}$

perfect square

$$= \left(x-\frac{7}{2}\right)\cdot\left(x-\frac{7}{2}\right)$$

O F I L

$$= x^2 - \frac{7}{2}\cdot x - \frac{7}{2}\cdot x + \boxed{\frac{7}{2}\cdot\frac{7}{2}}$$

F O I

$$= x^2 - \frac{\cancel{2}}{1}\cdot\frac{7}{\cancel{2}}x + \frac{49}{4}$$

$$= x^2 - 7x + \frac{49}{4}$$

$a=1$, $b=-7$, $c=\frac{49}{4}=12.25$

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$$(x - 3.5)^2 = (x - 3.5)\cdot(x - 3.5)$$

$$= x^2 - 3.5x - 3.5x + 12.25$$

$$= x^2 - 7x + 12.25$$

Note: $\boxed{3.5 \cdot 3.5} = \frac{35}{10} \cdot \frac{35}{10} = 12.25$

$$= \frac{5 \cdot 7 \cdot 5 \cdot 7}{100}$$

$$= \frac{5 \cdot 5 \cdot 7 \cdot 7}{100}$$

$$= \frac{25 \cdot 49}{100} = \frac{1225}{100}$$

$$49 \cdot 25 = (50 - 1) \cdot 25$$

$$= 50 \cdot 25 - 25$$

$$= 1250 - 25 = 1225$$

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4. For each of the problems above, write the equivalent expressions in the form

$$(x + d)^2 = x^2 + bx + c, \quad d = \frac{b}{2}, \quad c = d^2 = \left[\frac{b}{2}\right]^2$$

Then, specifically identify the values of the coefficients d, b and c. The first one is done for you.

$2d = b$

4A. $(x - 4)^2$

We notice from our work on problem 3A above that we have:

perfect square → $\boxed{(x-4)^2} = \boxed{x^2 - 8x + 16}$ trinomial $d = -4, b = -8, c = 16$

4B. $(x + 3)^2$

perfect square → $\boxed{(x+3)^2} = \boxed{x^2 + 6x + 9}$ trinomial

$d = 3$ $b = 6$, $c = 9$

4C. $(x + 11)^2$

perfect square → $\boxed{(x+11)^2} = x^2 + 22x + 121$ perfect-square trinomial

a base squared

$d = 11$ $b = 22$, $c = 121$

4D. $\left(x - \frac{7}{2}\right)^2$

$a = 1$

perfect square $\boxed{\left(x - \frac{7}{2}\right)^2} = x^2 - 7x + \frac{49}{4}$

$d = \frac{-7}{2}$ $b = -7$, $c = \frac{49}{4}$

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5. Look back on the work you finished in problem 4 above. What pattern do you notice? Specifically, how are the coefficients d, b and c related to each other? Make a conjecture about how this will work in general.

$$(x + d)^2 = x^2 + bx + c$$

Guess : $d = \frac{b}{2} \Leftrightarrow 2 \cdot d = b$

$$c = d^2 = \left[\frac{b}{2}\right]^2 = \frac{b^4}{4}$$

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6. Your definition of a perfect square trinomial

Homework

We say that a perfect-square trinomial is a three-term expression that can be factored as a perfect square. We've seen this in our work in problems 1 – 5. Below is a diagram that shows the connection:

$$\boxed{x^2 + bx + c} = \boxed{(x+d)^2}$$

perfect-square trinomial — perfect square

Come up with your own description for a perfect-square trinomial. Use simple, abuelita language and make this as clear as you can for yourself.

Nerdy language:

a perfect-square trinomial

Abuelita language:

- a math sentence that has three "parts" connected together by addition.
- each of the three parts has a very special role to play and also special anatomy...
- Lets look at each of the three parts or pieces individually:

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$$\boxed{x^2} + \boxed{bx} + \boxed{c}$$

first piece | second piece | third piece

$\boxed{\text{First piece :} \quad x^2}$

- This part includes the single variable x raised to the power of two
- In this form, the x^2 has a scalar of 1 in front which is hidden from view with

$$x^2 = 1 \cdot x^2$$

Second piece: bx

- This part includes two features: the number b and the variable x

- We can write this more formally as

$$bx = b \cdot x^1$$

where b is a constant and x is the input variable.

Third piece: c

- this part includes a constant value labeled as c with no variable

- we can say the variable is hidden an write

$$c = c \cdot x^0$$

since $x^0 = 1$.

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7. LEARN TO COMPLETE THE SQUARE

Consider each incomplete expression below. Add a constant to make the expression a perfect-square trinomial. Then write the factored form of the expression as a perfect square. Identify each step you take in the solution. Please make sure you can explain to yourself why you are taking each step.

☑ 7A. $x^2 + 10x$

perfect square trinomial

perfect-square

$$\boxed{x^2 + 10x + 25} = \boxed{(x + 5)^2}$$

$b = 10$ $c = d^2 = 5^2$ $d = \frac{b}{2} = \frac{10}{2} = 5$

Check: $(x+5)^2 = (x+5)\cdot(x+5)$

$= x^2 + 5x + 5x + 25$

$= x^2 + 10x + 25$

☐ 7B. $t^2 - 7t$

perfect square trinomial

perfect square

$$\boxed{t^2 + -7t + \frac{-49}{4}} = \boxed{\left(x + \frac{-7}{2}\right)^2}$$

$b = -7$ $c = d^2$ $d = \frac{b}{2} = \frac{-7}{2}$

$= \left[\frac{b}{2}\right]^2$

$= \left[\frac{-7}{2}\right]^2$

$= \frac{-7}{2} \cdot \frac{-7}{2}$

$= \frac{49}{4}$

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7C. $x^2 - \frac{11}{2}x$

perfect-square trinomial

perfect square

$$\boxed{x^2 + \frac{-11}{2}x + \frac{121}{4}} = \boxed{\left(x + \frac{-11}{4}\right)^2}$$

$b = \frac{-11}{2}$, $c = d^2 = \left[\frac{b}{2}\right]^2$

$= \left[\frac{-11}{2}\right]^2$

$= \frac{-11}{2} \cdot \frac{-11}{2}$

$= \frac{+121}{4}$

$d = b \div 2$

$= \frac{-11}{2} \div \frac{2}{1}$

$= \frac{-11}{2} \cdot \frac{1}{2}$

$= \frac{-11}{4}$

7D. $m^2 + \frac{5}{4}m$

perfect-square trinomial

perfect square

$$\boxed{m^2 + \frac{5}{4} \cdot m + \frac{25}{64}} = \boxed{\left(m + \frac{5}{8}\right)^2}$$

$b = \frac{5}{4}$

$c = d^2$

$= \left[\frac{5}{8}\right]^2$

$= \frac{5}{8} \cdot \frac{5}{8} = \frac{25}{64}$

$d = \frac{b}{2} = b \div 2$

$= \frac{5}{4} \div \frac{2}{1}$

reciprocal

$= \frac{5}{4} \cdot \frac{1}{2}$

$= \frac{5}{8}$

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7E. $5\,x^2 - 30\,x$

$$\underbrace{5x^2}_{} - 30x$$

we begin by factoring out the 5 so that the squared term has a 1 in front of x^2

$$= 5(x^2 - 6x)$$

$$= 5(x^2 - 6x + 9)$$

$b = -6$

$$c = \left[\frac{b}{2}\right]^2 = \left[\frac{-6}{2}\right]^2 = [-3]^2 = 9$$

$$= 5\cdot(x + -3)^2$$

$$d = \left[\frac{b}{2}\right] = \frac{-6}{2} = -3$$

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8. GENERATE THE VERTEX FORM FOR A QUADRATIC FUNCTION

Use the method of completing the square (from problems 1 – 7) to transform the quadratic function in standard form into an expression that contains a perfect square

8A. $f(x) = 5x^2 + 8x + 3$

$f(x) = 5x^2 + 8x + 3$

$= 5\left(x^2 + \frac{8}{5}x\right) + 3$

$b = \frac{8}{5}$

we want to add something special here to create a perfect-square trinomial inside the parenthesis

$= 5\left(x^2 + \frac{8}{5}x + \bigcirc\right) + 3$

Recall: from our previous work in this handout, we have

$x^2 + bx + c = (x+d)^2$

where $d = \frac{b}{2}$ and $c = d^2$

For our case with $b = \frac{8}{5}$ we see that

$d = b \div 2 = \frac{8}{5} \div \frac{2}{1}$

$\Rightarrow d = \frac{8}{5} \cdot \frac{1}{2}$

$\Rightarrow \boxed{d = \frac{8}{2 \cdot 5} = \frac{8}{10}}$

$\Rightarrow d^2 = \left[\frac{8}{2 \cdot 5}\right]^2 = \frac{8}{2 \cdot 5} \cdot \frac{8}{2 \cdot 5}$

$\Rightarrow \boxed{c = \frac{8^2}{4 \cdot 5^2} = \frac{64}{100}}$

8B. $f(x) = ax^2 + bx + c$

$f(x) = ax^2 + bx + c$

$= a\left(x^2 + \frac{b}{a}x\right) + c$

$= a\left(x^2 + \frac{b}{a}x + \bigcirc\right) + c$

For the more general case, we will write:

$d = \frac{b}{a} \div 2$

$\Rightarrow d = \frac{b}{a} \cdot \frac{1}{2}$

$\Rightarrow d = \frac{b}{2 \cdot a}$

$\Rightarrow d^2 = \left[\frac{b}{2 \cdot a}\right]^2 = \frac{b}{2 \cdot a} \cdot \frac{b}{2 \cdot a}$

$\Rightarrow d^2 = \frac{b^2}{4 \cdot a^2}$

$$\Rightarrow f(x) = 5\left(x^2 + \frac{8}{5}x + 0\right) + 3$$

$$= 5\left(x^2 + \frac{8}{5}x + \underbrace{\frac{64}{100} - \frac{64}{100}}_{\text{this equals zero}}\right) + 3$$

we can distribute

$$= 5\cdot\left(\underbrace{x^2 + \frac{8}{5}x + \frac{64}{100}}_{\text{here is the perfect-square trinomial}} - \frac{64}{100}\right) + 3$$

$$= 5\cdot\left(x^2 + \frac{8}{5}x + \frac{64}{100}\right) - \frac{64}{20} + 3$$

$$= 5\left(\underbrace{x^2 + \frac{8}{5}x + \frac{64}{100}}_{\text{this can be factored}}\right) + 3 - \frac{64}{20}$$

$$= 5\cdot\left(x + \frac{8}{10}\right)^2 + \left[3 - \frac{64}{20}\right]$$

$$= a\left(x - \frac{-8}{10}\right)^2 + \left[3 - \frac{64}{20}\right]$$

$$\Rightarrow f(x) = a\cdot\left(x^2 + \frac{b}{a}x + 0\right) + c$$

$$= a\cdot\left(x^2 + \frac{b}{a}x + \underbrace{\frac{b^2}{4\cdot a^2} - \frac{b^2}{4\cdot a^2}}_{\text{this equals zero}}\right) + c$$

$$= a\left(\underbrace{x^2 + \frac{b}{a}x + \frac{b^2}{4\cdot a^2}}_{\text{here is a perfect-square trinomial}} - \frac{b^2}{4\cdot a^2}\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4\cdot a^2}\right) - \frac{b^2}{4\cdot a} + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4\cdot a^2}\right) + c - \frac{b^2}{4\cdot a}$$

$$= a\left(x + \frac{b}{2\cdot a}\right)^2 + \left[c - \frac{b^2}{4a}\right]$$

$$= a\left(x - \frac{-b}{2a}\right)^2 + \left[c - \frac{b^2}{4a}\right]$$

$$= a\cdot(x - h)^2 + k$$